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Bachelorarbeit

Variance of photon mapping density estimation

Zhiming Gan



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Aufgabensteller: Prof. Dr. Dieter Kranzlmüller
Betreuer: Dr. Rubén Jesús García Hernández
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München, den 11. Dezember 2018

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(Unterschrift des Kandidaten)

Abstract

Computer graphics is a sub-field of computer science which studies methods for digitally synthesizing and manipulating visual content. The major subfields in computer graphics might be geometry, animation, imaging, topology and rendering. The rendering generates images from a model. It may simulate light transport to create realistic images or it may create non-photorealistic images that have a particular artistic style. The rendering techniques can be classified into local illumination techniques and global illumination techniques. The local illumination algorithms are very fast, but images rendered using global illumination algorithms often appear more photorealistic than those using only local illumination algorithms. There are many algorithms used in global illumination. Photon mapping is a very popular two-pass global illumination algorithm: the first pass is casting photons from the light source and saving the information of reflection in the so called photon map; in the second pass, the brightness of the pixels are estimated from the photon map, this pass is also called radiance estimate.

Many studies have successfully tried to improve the radiance estimation, for example, the radiance estimation can be improved by adding a filter of the distance. Because the error analysis of rendering algorithms is beneficial for understanding their behavior, this thesis extended the framework presented by Dr. García and presents an analysis of the variance in common used filtering kernels in the context of photon mapping density estimation. We use the joint distribution of order statistics to calculate the variance value in both 2D and 3D case for the constant kernel, the epanechnikov kernel, the silverman kernel (also called quartic kernel), the cone filter and the triangle kernel, which is a special case of the cone filter. Corresponding to our theoretical study, we have implemented a scene consisting of a planar unit disc, a very simple model, which illuminated by a directional light source for each kernel and calculated the signal to noise ratio of each kernel. After it we show the signal-to-noise ratio of our theoretical results and its computational cost compares to the empirical results and computational time. The theoretical comparison among different filters allow us to choose the best kernel for our needs. Besides, we could offer a threshold, the estimation of computational cost of algorithms which stop after error is smaller than the threshold.

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1. Introduction

In this thesis, we have extended the framework presented in García et al. [GHUPS14] and used the joint distribution of order statistics to calculate the variance value of photon mapping density estimation in common used filtering kernels, because the error analysis of rendering algorithms is beneficial for understanding their behavior [ATS94] and the radiance estimation can be improved by adding a filter of the distance [Zuc17],

Section 1.1 to 1.5 introduces the background knowledge like computer graphics, rendering, local illumination, global illumination and photon mapping algorithm. Section 1.6 gives a short survey of the related previous works. Section 1.7 introduces the mathematical and statistical prerequisites that we have used in our study, with 1.7.1 introducing integral, 1.7.2 introducing variance, section 1.7.3 introducing order statistics and section 1.8 applying order statistics to photon mapping density estimation in both 2D case and 3D case. Then section 2 describes in detail our theoretical study of the variance value of the irradiance estimate for the constant kernel, the epanechnikov kernel [Wal98], the silverman/quartic kernel [WMHSPG98], the cone filter [Jen96a] inclusive triangle kernel and the gaussian kernel [Sch03]. Section 3 contains an empirical study of the kernels using a directional light source. Section 4 provides the conclusions and our near future work plan.

1.1. Computer graphics

Computer graphics [SM09] is a science of using mathematical algorithms to convert two- or three-dimensional graphics into a raster form of a computer display. Generally, the main research content of computer graphics is to study how to represent graphics in computers, and the related principles and algorithms for computing, processing and displaying graphics using computers.

The main goal of computer graphics is to create effective visual communication. In the field of science, graphics can show scientific results to the public in a visual way; in the field of entertainment, such as PC games, mobile games, 3D movies and film special effects, computer graphics plays an increasingly important role; Graphics is also playing an important role in creative or artistic creation, commercial advertising, product design and other industries. In the field of science, this was highlighted in the report Visualization in Scientific Computing in 1987 [McC88]. The report cites Richard Hamming's classic assertion in 1962 [Ham86]: "The purpose of computing is insight, not numbers." The report mentions the important role of computer graphics in helping the human brain understand the essence of things from the perspective of graphical images, because graphic images have more insight than simple numbers.

With the continuous development of computer graphics, its application range has become increasingly widespread. The main application areas of computer graphics are computer-aided design and computer-aided manufacturing (CAD/CAM), Computer aided instruction (CAI), Computer animation, Management and office automation, Scientific computing visualization, Computer game and Virtual Reality (VR).

1. Introduction

Computer graphics mainly consists of four parts: Modeling, Rendering, Animation, and Human-computer Interaction (HCI).

To represent a three-dimensional object in a computer, you must first have its geometric model representation. Therefore, the modeling of 3D models is the basis of computer graphics and the premise of other content. Expressing a geometric object can be expressed by a mathematical spline function or an implicit function; It can also be expressed by a triangular mesh expressed by sampling points on a smooth surface and its connection relationship. In figure 1.1, we have an example of a triangle mesh representing a dolphin.

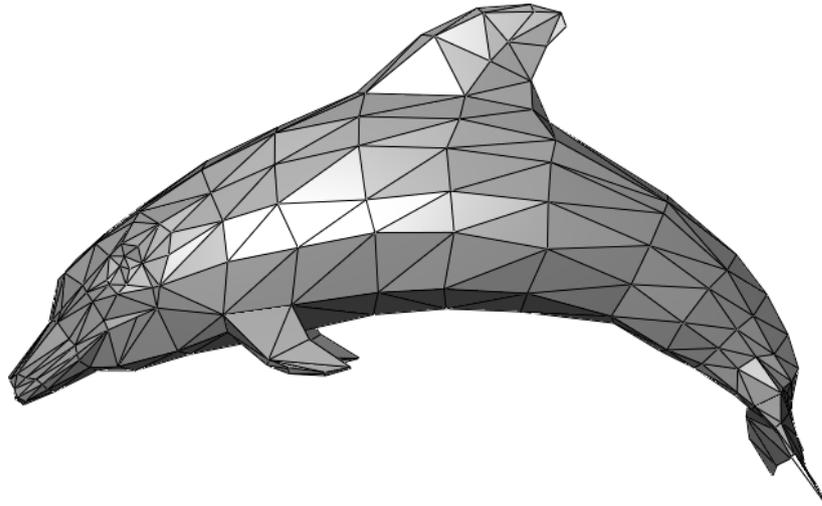


Figure 1.1.: Example of a triangle mesh representing a dolphin. Picture with permission from author: Chrschn

With 3D models or scenes, how do you draw these 3D geometric models to produce a pleasing and realistic image? This is the main task of traditional computer graphics. In computer-aided design, film and television animation and various visual applications, high requirements for the high-fidelity of graphics rendering results are put forward.

Animation is the effect of motion of an object by continuously playing a still image. Computer animation is one of the research hotspots of computer graphics by using programming or animation software to generate a series of scene images. Its research interests include human animation, motion animation, script animation, and animation systems for human characters. In addition, the dynamic simulation of highly physical reality and the real simulation of various physical phenomena are also major issues in the field of animation.

Human-Computer Interaction (HCI) refers to the process of information exchange between a person and a computer that determines the task in a certain interaction or interaction interface between the person and the computer. Generally, it is how people tell the computer through certain interactions to accomplish the tasks he wants to accomplish. In recent years, human-computer interaction devices have undergone tremendous development, and various natural interaction methods have emerged in an endless stream, greatly enriching the user-machine interaction experience, facilitating user operations, and easily expressing user interaction intentions. There are many interesting human-computer interaction devices, such as Google Glass, 3D Printer, Apple iWatch, etc.

1.2. Rendering

Rendering or image synthesis [AMHH08] is the automatic process of generating a photo-realistic [Car15] or non-photorealistic [SS02] image from a 2D or 3D model by means of computer programs. Also, the results of displaying such a model can be called a render.

Rendering has uses in architecture, video games, simulators, movie or TV visual effects, and design visualization, each employing a different balance of features and techniques. The rendering process has the following steps:

1. You must locate the camera in a 3D scene, which is the same as real photography. In general, 3D software has provided four default cameras, which are the four main windows in the software, divided into top view, front view, side view and perspective. Most of the time we render perspectives rather than other views. The cameras in perspective basically follow the principle of real cameras, so the results we see will be as stereoscopic as the real 3D world.
2. In order to reflect the sense of space, the rendering program has to do some special work, which is to decide which objects are in front, which objects are behind and which objects are occluded. The sense of space is not perfectly reproduced by the occlusion relationship of objects. Many beginners in three dimensions only pay attention to the shaping of the three-dimensional sense and ignore the sense of space. It is necessary to know that the sense of space and the attenuation of the light source, the environment fog, and the depth of field effect are closely related.
3. After the rendering program has acquired the range that needs to be rendered by the camera, it is necessary to calculate the influence of the light source on the object, which is the same as the real world. Many 3D software have a default light source, otherwise we can't see the coloring effect in the perspective, especially rendering. Therefore, the rendering program is to calculate the effect of each light source we add in the scene on the object. Unlike real-world sources, rendering programs often compute a large number of auxiliary sources. In the scene, some light sources illuminate all objects, and some light sources only illuminate an object, which complicates the simple things.
4. To use depth map shadows or raytraced shadows? This often depends on whether a transparent material object is used in the scene to calculate the shadow cast by the light source. In addition, after using the area light source, the renderer also calculates a special shadow, the soft shadow, If the light source in the scene uses a light effect, the rendering program will also spend more system resources to calculate the results of the special effects, especially the volumetric light, also known as the light fog. It takes up a lot of system resources and must be used when using it.
5. The rendering program also calculates the color of the surface of the object according to the material of the object. The type of the material is different, the properties are different, and the textures will produce different effects. Moreover, this result does not exist independently, it must be combined with the aforementioned light source. If there are particle systems in the scene, such as flames, smoke, etc., the rendering program should be considered.

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Many rendering algorithms have been researched, and software used for rendering may employ a number of different techniques to obtain a final image. Today's rendering technology has been able to render a variety of objects, including skin, trees, flowers, water, smoke, hair, etc., They are very realistic. Some commercial software (such as Maya, Blender, Pov Ray, etc.) also provide powerful realistic rendering capabilities, which are often used in computer graphics research papers to render beautiful presentations or result graphs. In figure 1.2, it is an image created by using Pov Ray 3.6.



Figure 1.2.: An image created by using POV-Ray 3.6. Picture with permission from author: Gilles Tran

The rendering technique can be classified into local illumination technique and global illumination technique.

1.3. Local illumination

The local illumination or direct illumination is the lighting (shading) method that was originally used by computer graphics programs. When rendering, direct illumination considers only light from the original source, no bounce light or light emissive polygons (or sources other than lights) will add to the lighting within the rendered scene.

The local illumination is very fast, and often used for real-time applications such as 3D computer games. But the images synthesized appear artificial and it requires the user to add the effects like shadows and mirrors one by one.

There are many popular local illumination methods such as Phong reflection, Lambert's model and Gouraud shading.

The Phong reflection model (also called Phong illumination or Phong lighting), it is an empirical model of the local illumination of points on a surface. The Phong reflection model was developed by Bui Tuong Phong [Pho75].

Lambert's model [ON94] for body reflection is widely used in computer graphics. It is used

extensively by rendering techniques such as radiosity and ray tracing. For several real-world objects, however, Lambert’s model can prove to be a very inaccurate approximation to the body reflectance. While the brightness of a Lambertian surface is independent of viewing direction, that of a rough surface increases as the viewing direction approaches the light source direction.

Gouraud shading [Gou71] named after Henri Gouraud, is an interpolation method used in computer graphics to produce continuous shading of surfaces represented by polygon meshes. In practice, Gouraud shading is most often used to achieve continuous lighting on triangle surfaces by computing the lighting at the corners of each triangle and linearly interpolating the resulting colours for each pixel covered by the triangle. Gouraud shading works as follows: An estimate to the surface normal of each vertex in a polygonal 3D model is either specified for each vertex or found by averaging the surface normals of the polygons that meet at each vertex. Using these estimates, lighting computations based on a reflection model, e.g. the Phong reflection model, is then performed to produce colour intensities at the vertices. For each screen pixel that is covered by the polygonal mesh, colour intensities can then be interpolated from the colour values calculated at the vertices.

1.4. Global illumination

Global illumination [DBBS06] or indirect illumination, is a general name for a group of algorithms used in 3D computer graphics that are meant to add more realistic lighting to 3D scenes. Such algorithms take into account not only the light that comes directly from a light source (direct illumination), but also subsequent cases in which light rays from the same source are reflected by other surfaces in the scene, whether reflective or not (indirect illumination).

Images rendered using global illumination algorithms often appear more photorealistic than those using only direct illumination algorithms. However, such images are computationally more expensive and consequently much slower to generate.

There are many popular global illumination methods such as radiosity [SP94], ray tracing [Gla89], monte carlo ray tracing [MP01] and photon mapping [Jen96a].

Radiosity methods were first developed in about 1950 in the engineering field of heat transfer. They were later refined specifically for the problem of rendering computer graphics in 1984 [GTGB84].

The first ray tracing algorithm used for rendering was presented in 1968 [App68]. This algorithm has since been termed “ray casting” . The idea behind ray casting is to shoot rays from the eye, one per pixel, and find the closest object blocking the path of that ray.

The monte carlo ray tracing is an approach that simulates the light reflection at diffuse surfaces, it can simulate indirect lighting. In addition one might add that monte carlo ray tracing methods can be very easy to implement. The main problem with monte carlo ray tracing is variance seen as noise in the rendered images. It requires a lot of samples per pixel to reduce the noise.

The photon mapping is a two-pass global illumination algorithm developed by Henrik Wann Jensen that approximately solves the rendering equation [Kaj86].

In figure 1.3 we have two pictures, the left one is created by local illumination method, the right one by global illumination method, we can see that the right one is more realistic.

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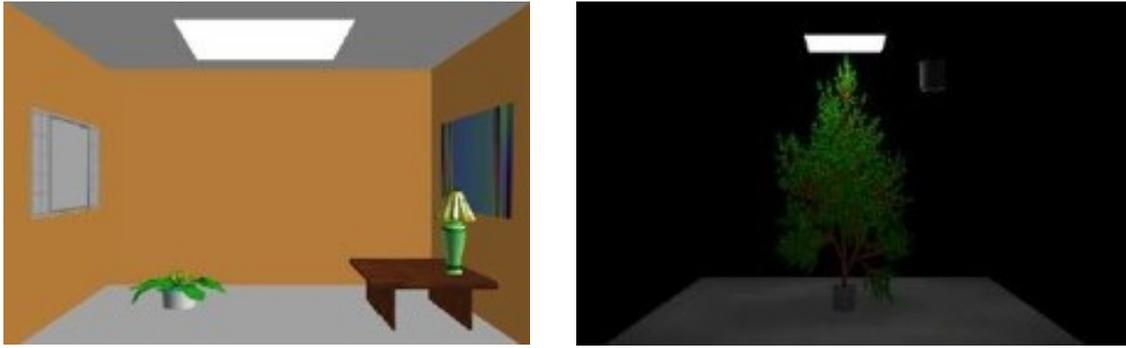


Figure 1.3.: Pictures created by local illumination method (left) and global illumination method (right). Picture with permission from author: Dr. García

1.5. Photon mapping

Photon mapping [Jen96a] can simulate global illumination effects including diffuse reflection, caustics, etc. It's as flexible as general monte carlo ray tracing [JC94], but more efficiently. The method is to emit radiation from the light source and the camera respectively, and when the termination condition is satisfied, the two rays are connected together to generate a radiance value in the next step.

The photon mapping algorithm includes several computation steps:

Photon emission and tracking, creating photon maps: First, a certain number of photons are emitted from the light source to the scene, and the direction of each photon is randomly generated, and the power of the light source is uniform distributed to each emitted photon. When a photon hits an object, it may be reflected, refracted or absorbed. This is determined by the material properties of the surface of the object, usually using the Russian roulette method. When a photon hits the diffuse surface in the scene, the information of the photon is stored in the global photon map. If the photon is specularly reflected or refracted before hitting the diffuse surface, the photon is stored in the caustic photon. The main problem with photon mapping is finding the k nearest photons to the sampling point in a large number of photons. In order to quickly find photons, a balanced kd-tree is usually used to store photons.

Radiance estimation and rendering: Through the photon map generated in the first step, the k nearest photons to the sampling point are searched, and the radiance of the sampling point (ie, the indirect illumination value) is estimated according to the found power. This process is called density estimation. If you want to get high quality rendering, you need to use the final Gathering instead of the density estimate. But the cost of this calculation is very large

The contribution of the k nearest photons is commonly weighted by a filtering kernel. such as the cone filter [Jen96a], the epanechnikov kernel [Wal98] and the silverman kernel [WMHSPG98]. The gaussian kernel and the specialized differential filter have also been used, along with other filtering strategies [Sch03].

A weight depending on the distance to the point where radiance is being estimated is often used to provide smoothing. This avoids artifacts when the number of photons is small. The

weight function is called a kernel. If no weights are used, the result is equivalent to using a constant kernel. This procedure is essentially a kernel density estimation technique [WJ94].

1.6. Survey of the previous work

The role of global illumination algorithms is to simulate the interaction of light with large-scale geometry for the purpose of image synthesis. The greatest challenges in this endeavor have been those of accuracy and efficiency. While issues of efficiency have been addressed frequently in computer graphics, error analysis has received comparatively little attention. [ATS94]

Arvo has attempted to establish a framework for error analysis within a well-defined class of global illumination problems. [ATS94]

Ashikhmin has attempted to prove the convergence properties of the MLT (Metropolis Light Transpo) algorithm. This does not by itself yield immediate practical implications, but some of the tools used in our proofs may eventually prove to be useful for error estimation and reduction [SSS01]

The photon mapping method uses k nearest neighbors to compute the outgoing radiance at a shading point. The radiance estimation can be improved by adding a filter of the distance. With kernels, we can get a smoother result, the MISE (Mean integrated squared error) can also be minimized with respect to the kernel used.

In Jensen [Jen96a], a filter was added to weigh each photon according to its distance to the shading point, it is expressed as the cone filter.

Another useful filter is the epanechnikov kernel, which can reduce the mean integrated square error of the kernel density estimation. In photon mapping, Roland [Sch03] employed it and achieved smooth results.

In García et al. [GHUS12] was shown that the photon mapping algorithm with constant kernel should discard the contribution of the k^{th} nearest photon in order to avoid an overestimation bias.

In García et al. [GHUPS14] it was shown that the estimator provided by the cone filter is not consistent unless the slope is one (yielding the triangular kernel), and that the epanechnikov and silverman/quartic kernels are consistent. The gaussian filter has two different estimation biases.

1.7. Mathematical and statistical prerequisites

First of all, we will introduce the mathematical and statistical prerequisites that we have used in this thesis for our study.

1.7.1. Integral

An integral [LS14] assigns numbers to functions in a way that can describe displacement, area, volume, and other concepts that arise by combining infinitesimal data.

Given a function f of a real variable x and an interval $[a, b]$ of the real line, the definite integral is:

$$\int_a^b f(x)dx \tag{1.1}$$

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if f is a continuous real-valued function defined on a closed interval $[a, b]$, then, once an antiderivative F of f is known, the definite integral of f over that interval is given by:

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a) \quad (1.2)$$

with the following properties:

$$\int_a^b (\alpha f + \beta g)(x)dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx \quad (1.3)$$

In our study we have to evaluate double integrals and triple integrals.

If T is a domain that is normal with respect to the xy -plane and determined by the functions $\alpha(x, y)$ and $\beta(x, y)$, then:

$$\iiint_T f(x, y, z)dx dy dz = \iint_D \int_{\alpha(x, y)}^{\beta(x, y)} f(x, y, z)dx dy dz \quad (1.4)$$

To compute a double integral we could use the Fubini's theorem [Jon01]. If $f(x, y)$ is $X \times Y$ integrable, it means that it is measurable and

$$\int_{X \times Y} |f(x, y)|d(x, y) < \infty \quad (1.5)$$

then

$$\begin{aligned} \int_X \left(\int_Y f(x, y)dy \right) dx &= \int_Y \left(\int_X f(x, y)dx \right) dy \\ &= \int_{X \times Y} f(x, y)d(x, y) \end{aligned} \quad (1.6)$$

1.7.2. Variance

Variance [Fel68] is an important tool in the sciences, where statistical analysis of data is common. It measures how far a set of (random) numbers are spread out from their average value.

The variance of a random variable X is the expected value of the squared deviation from the mean of X , $\mu = E[X]$.

$$Var[X] = E[(X - \mu)^2] \quad (1.7)$$

The expression for the variance can be expanded:

$$Var[X] = E[X^2] - (E[X])^2 \quad (1.8)$$

If the random variable X represents samples generated by a continuous distribution with probability density function $f(x)$, and $F(x)$ is the corresponding cumulative distribution function, then the population variance is given by:

$$\begin{aligned} Var[X] &= \int (x - \mu)^2 f(x)dx \\ &= \int x^2 f(x)dx - \mu^2 \end{aligned} \quad (1.9)$$

where μ is the expected value of X given by $\mu = E(X)$.

For the sum of N random variables $\{X_1, \dots, X_N\}$:

$$\begin{aligned} \text{Var} \left[\sum_{i=1}^N X_i \right] &= \sum_{i,j=1}^N \text{Cov}[X_i, X_j] \\ &= \sum_{i=1}^N \text{Var}[X_i] + \sum_{i \neq j} \text{Cov}[X_i, X_j] \\ &= \sum_{i=1}^N \text{Var}[X_i] + 2 \sum_{1 \leq i < j \leq N} \text{Cov}[X_i, X_j] \end{aligned} \quad (1.10)$$

Where $\text{Cov}[\]$ is the covariance:

$$\text{Cov}[X, Y] = E[XY] - E[X]E[Y] \quad (1.11)$$

The formula 1.10 states that the variance of a sum is equal to the sum of all elements in the covariance matrix of the components. The last expression states equivalently that the variance of the sum is the sum of the diagonal of covariance matrix plus two times the sum of its upper triangular elements.

Together we get:

$$\text{Var} \left[\sum_{i=1}^N X_i \right] = \sum_{i=1}^N \text{Var}[X_i] + 2 \sum_{1 \leq i < j \leq N} (E[X_i X_j] - E[X_i]E[X_j]) \quad (1.12)$$

1.7.3. Order Statistics

Order statistics [DN04] deal with the probability distribution of elements in ordered lists. When we sample a continuous random phenomenon which has a probability distribution f , the result is a list of real values $[x_1, \dots, x_n]$. After ordering this list, we obtain $[x_{(1)}, \dots, x_{(n)}]$, where $x_{(1)}$ is the minimum value, and $x_{(n)}$ the maximum.

The probability distribution of the i th order statistic is:

$$f_{X_{(i)}}(x) = \mathcal{C} F^{i-1}(x) (1 - F(x))^{n-i} f(x) \quad (1.13)$$

where F is the cumulative distribution function corresponding to f and

$$\mathcal{C} = \frac{n!}{(k-1)!(n-k)!} \quad (1.14)$$

The joint distribution of two order statistics is:

$$f_{X_{(i,k)}}(x, y) = \mathcal{C}_J [F(x)]^{i-1} f(x) [\mathcal{G}(x, y)]^{k-i-1} f(y) [\mathcal{F}(y)]^{n-k} \quad (1.15)$$

with

$$\mathcal{C}_J = \frac{n!}{(i-1)!(k-i-1)!(n-k)!} \quad (1.16)$$

and

$$\mathcal{F}(x) = 1 - F(x) \quad (1.17)$$

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$$\mathcal{G}(x, y) = F(y) - F(x) \quad (1.18)$$

In general, the joint distribution of k order statistics $X_{(n_1)}, \dots, X_{(n_k)}$, ($1 \leq n_1 < \dots < n_k \leq n$) is for $x_1 \leq \dots \leq x_k$:

$$f_{(n_1, \dots, n_k)} = n! \left[\prod_{j=1}^k f(x_j) \right] \prod_{j=0}^k \left(\frac{[F(x_{j+1}) - F(x_j)]^{n_{j+1} - n_j - 1}}{(n_{j+1} - n_j - 1)!} \right) \quad (1.19)$$

where $x_0 = -\infty$, $x_{k+1} = +\infty$, $n_0 = 0$ and $n_{k+1} = n + 1$.

The joint distribution of three order statistics is then:

$$f_{i,j,k}(x, y, z) = n! f(x) f(y) f(z) \frac{[F(x)]^{i-1}}{(i-1)!} \frac{[F(y) - F(x)]^{j-i-1}}{(j-i-1)!} \frac{[F(z) - F(y)]^{k-j-1}}{(k-j-1)!} \frac{[1 - F(z)]^{n-k}}{(n-k)!} \quad (1.20)$$

for $x \leq y \leq z$ and 0 otherwise.

1.8. Application of Order Statistics to Photon Mapping

We will now applicate order statistics to photon mapping with uniform rediance in 2D [GHUS12] [GHUPS14] and 3D cases.

Uniform density of impacts

We study the expected value of photon mapping estimators for a unit disc with uniform radiance. We assume all photons carry the same flux $\phi = \frac{\pi I(P)}{(n)}$. The location of impacts has a constant distribution function:

$$g(P) = \frac{1}{\pi} \quad (1.21)$$

with P being a point in the unit disc. The probability density of photons at distance r and the cummulative distributions are:

$$f(r) = 2r \quad (1.22)$$

and

$$F(r) = r^2 \quad (1.23)$$

The distribution of the k^{th} order statistic (the distance of the impact sought by Photon Maps) is

$$f_{X_{(k)}}(r) = \mathcal{C} F(r)^{k-1} (1 - F(r))^{n-k} f(r) \quad (1.24)$$

Substituting, we get

$$f_{X_{(k)}}(r) = \mathcal{C} 2r^{2k-1} (1 - r^2)^{n-k} \quad (1.25)$$

The joint distribution is 1.15

$$\begin{aligned} f_{X_{(i,k)}}(r_i, r_k) &= \mathcal{C}_J [F(r_i)]^{i-1} f(r_i) [\mathcal{G}(r_i, r_k)]^{k-i-1} f(r_k) [\mathcal{F}(r_k)]^{n-k} \\ &= \mathcal{C}_K 2r_i^{2i-1} [r_k^2 - r_i^2]^{k-i-1} \end{aligned} \quad (1.26)$$

with (notice there was a typo in the definition of $\mathcal{F}(r_k)$ in García et al. [GHUPS14], in which r_k should be squared)

$$\mathcal{G}(r_i, r_k) = F(r_k) - F(r_i) = r_k^2 - r_i^2 \quad (1.27)$$

$$\mathcal{F}(r_k) = 1 - F(r_k) = 1 - r_k^2 \quad (1.28)$$

and we define \mathcal{C}_K as

$$\mathcal{C}_K = 2C_J r_k [1 - r_k^2]^{n-k} \quad (1.29)$$

The joint distribution of three order statistics is 1.20.

$$\begin{aligned} f_{X_{(i,j,k)}}(r_i, r_j, r_k) &= \frac{n! 2r_i 2r_j 2r_k [r_i^2]^{i-1} [r_j^2 - r_i^2]^{j-i-1} [r_k^2 - r_j^2]^{k-j-1} [1 - r_k^2]^{n-k}}{(i-1)!(j-i-1)!(k-j-1)!(n-k)!} \\ &= \mathcal{C}_{\textcircled{1}} 2r_i 2r_j 2r_k [r_i^2]^{i-1} [r_j^2 - r_i^2]^{j-i-1} [r_k^2 - r_j^2]^{k-j-1} [1 - r_k^2]^{n-k} \end{aligned} \quad (1.30)$$

and we define $\mathcal{C}_{\textcircled{1}}$ as

$$\mathcal{C}_{\textcircled{1}} = \frac{n!}{(i-1)!(j-i-1)!(k-j-1)!(n-k)!} \quad (1.31)$$

Volumetric effects in photon mapping

We will study here the case of an homogeneous, isotropic sphere of unit radius. In this case, the probability density of the photon interactions is constant in the sphere.

$$g(v) = \frac{1}{\frac{4}{3}\pi} = \frac{3}{4\pi} \quad (1.32)$$

for the v inside the sphere, and $g(v) = 0$ otherwise.

If we denote the power contained in the sphere by W and the volume of the sphere by V , the Power density is:

$$PD = \frac{W}{V} = \frac{3W}{4\pi} \quad (1.33)$$

If all photons carry the same power ϕ ,

$$\phi = \frac{W}{n} = \frac{4\pi PD}{3n} \quad (1.34)$$

The cumulative distributions and the probability density of photons at distance r are:

$$F(r) = \frac{4}{3}\pi r^3 \frac{3}{4\pi} = r^3 \quad (1.35)$$

and

$$f(r) = 3r^2 \quad (1.36)$$

The distribution of the k^{th} order statistic (the distance of the impact sought by Photon Maps) is

$$f_{X_{(k)}}(r) = \mathcal{C} F(r)^{k-1} (1 - F(r))^{n-k} f(r) \quad (1.37)$$

1. Introduction

Substituting, we get

$$\begin{aligned} f_{X_{(k)}}(r) &= \mathcal{C}(r^3)^{k-1}(1-r^3)^{n-k}3r^2 \\ &= \mathcal{C}3r^{3k-1}(1-r^3)^{n-k} \end{aligned} \quad (1.38)$$

The joint distribution is 1.15

$$\begin{aligned} f_{X_{(i,k)}}(r_i, r_k) &= \mathcal{C}_J[F(r_i)]^{i-1}f(r_i)[\mathcal{G}(r_i, r_k)]^{k-i-1}f(r_k)[\mathcal{F}(r_k)]^{n-k} \\ &= \mathcal{C}_J[(r_i^3)^{i-1}3(r_i)^2[\mathcal{G}(r_i, r_k)]^{k-i-1}3(r_k)^2[\mathcal{F}(r_k)]^{n-k} \\ &= \mathcal{C}_J3r_i^{3i-1}[\mathcal{G}(r_i, r_k)]^{k-i-1}3r_k^2[\mathcal{F}(r_k)]^{n-k} \end{aligned} \quad (1.39)$$

with

$$\mathcal{G}(r_i, r_k) = F(r_k) - F(r_i) = r_k^3 - r_i^3 \quad (1.40)$$

$$\mathcal{F}(r_k) = 1 - F(r_k) = 1 - r_k^3 \quad (1.41)$$

The joint distribution of three order statistics is 1.20.

$$\begin{aligned} f_{X_{(i,j,k)}}(r_i, r_j, r_k) &= \frac{n!3r_i^23r_j^23r_k^2[r_i^3]^{i-1}[r_j^3 - r_i^3]^{j-i-1}[r_k^3 - r_j^3]^{k-j-1}[1 - r_k^3]^{n-k}}{(i-1)!(j-i-1)!(k-j-1)!(n-k)!} \\ &= \mathcal{C}_{\textcircled{3}}3r_i^23r_j^23r_k^2[r_i^3]^{i-1}[r_j^3 - r_i^3]^{j-i-1}[r_k^3 - r_j^3]^{k-j-1}[1 - r_k^3]^{n-k} \end{aligned} \quad (1.42)$$

with $\mathcal{C}_{\textcircled{3}}$ defined as in 1.31

2. Theoretical study

We will present here the basis for the study of filtering kernels in photon mapping. The symbols used repeatedly in the thesis can be seen in 2.1 and 2.2. Because we have extended the framework from Dr. García, for consistency reasons, here we follow the symbols that are used in García et al. [GHUPS14]

Photon mapping parameters	
P	Point where a photon map query is done.
n	Number of photon impacts.
k	Number of photons sought in a Photon Maps query. $k \geq 3$, otherwise the variance value is infinity.
r_k	Distance of the k^{th} nearest photon in a query. r_k follows the distribution $f_{X_{(k)}}$.
ϕ	Flux.
s	Slope of the cone filter.
Distribution functions	
g	Area density of photon impacts.
f	Area density at a given distance of the photon map query.
F	Accumulated distribution of f .
$f_{X_{(k)}}$	k^{th} order statistics of f .
$f_{X_{(i,k)}}$	Joint distribution of the i^{th} and k^{th} order statistics of f .
Irradiance	
$K_{r_k}(r_i)$	Filtering kernel.
I	Irradiance.
\hat{I}	Photon mapping irradiance estimate..
\hat{I}_i^*, \hat{I}_k^*	Estimate of the contribution of the i^{th} (respectively k^{th}) nearest photon to \hat{I} .
Power density	
PD	Power density.
$\widehat{PD}(r_k)$	Estimate of the power density by taking the k nearest photons.
Expected values and variance	
$E[]$	Expected value.
E_{k-1}	$\sum_{n=0}^N E[\hat{I}_i^*]$.
$Var[]$	Variance.
$Cov[]$	Covariance.

Figure 2.1.: Symbols used in this article

2. Theoretical study

Combinatorials in order statistics	
$\mathcal{C} = \frac{n!}{(k-1)!(n-k)!}$	Constant in univariate case.
$\mathcal{C}_J = \frac{n!}{(i-1)!(k-i-1)!(n-k)!}$	Joint distribution constant (i, k) .
$\mathcal{C}_{\textcircled{J}} = \frac{n!}{(i-1)!(j-i-1)!(k-j-1)!(n-k)!}$	Joint distribution constant (i, j, k) .
$\mathcal{C}_K = 2C_J r_k [1 - r_k^2]^{n-k}$	Auxiliary combinatorial.

Figure 2.2.: Combinatorials in order statistics

2.1. Constant kernel/No filter

If no kernels are used, the result is equivalent to using a constant kernel. So when no filter is used, we call it constant kernel in our thesis. First we will deal with the constant kernel. For constant kernel, we could calculate the variance value with the formula 1.9

$$\text{Var}[X] = \int x^2 f(x) dx - E[X]^2 \quad (2.1)$$

2.1.1. Original PM with constant kernel

First we will deal with the original photon mapping with constant kernel.

2.1.1.1. Two-dimensional

For the original PM with constant kernel, the estimate of irradiance depends on the distance of the k^{th} nearest photon, which we will denote r_k .

$$\hat{I}(r_k) = \frac{k\phi}{\pi r_k^2} = \frac{kI(P)}{n r_k^2} \quad (2.2)$$

The expectation of the estimate of irradiance in the photon mapping method is [GHUS12]:

$$E[\hat{I}(r_k)] = \frac{k}{k-1} I(P) \quad (2.3)$$

The variance value is:

$$\begin{aligned} \text{Var}[\hat{I}] &= \int_0^1 \hat{I}^2(r_k) f_{X(k)}(r_k) dr_k - \frac{k^2}{(k-1)^2} I^2(P) \\ &= \frac{k^2(n-k+1)I^2(P)}{(k-1)^2(k-2)n} \end{aligned} \quad (2.4)$$

From the formula 2.4, we can see that the variance $\text{Var}[\hat{I}] = \infty$ if $k = 2$, because the denominator would be then 0.

2.1.1.2. Three-dimensional

The estimate of power density:

$$\widehat{PD}(r_k) = \frac{k\phi}{\frac{4}{3}\pi r_k^3} = \frac{kPD}{nr_k^3} \quad (2.5)$$

The expectation of the estimate of power density is [GHUS12]:

$$E[\widehat{PD}(r_k)] = \frac{k}{k-1}PD \quad (2.6)$$

The variance value is:

$$\begin{aligned} Var[\widehat{PD}(r_k)] &= \int_0^1 \widehat{PD}^2(r_k) f_{X(k)}(r_k) dr_k - \frac{k^2}{(k-1)^2} PD^2 \\ &= \frac{k^2(n-k+1)PD^2}{(k-1)^2(k-2)n} \end{aligned} \quad (2.7)$$

The same as the 2D case, we can see here that the variance $Var[\hat{I}] = \infty$ if $k = 2$.

2.1.2. Corrected PM with constant kernel

Now we deal with the corrected photon mapping with constant kernel from García et al. [GHUS12]

2.1.2.1. Two-dimensional

The corrected irradiance estimate is:

$$\hat{I}(r_k) = \frac{(k-1)\phi}{\pi r_k^2} = \frac{(k-1)I(P)}{nr_k^2} \quad (2.8)$$

and the expected value is [GHUS12]:

$$E[\hat{I}(r_k)] = I(P) \quad (2.9)$$

The variance value is:

$$\begin{aligned} Var[\hat{I}] &= \int_0^1 \hat{I}^2(r_k) f_{X(k)}(r_k) dr_k - I^2(P) \\ &= I^2(P) \frac{n-k+1}{n(k-2)} \end{aligned} \quad (2.10)$$

Again, from the formula 2.10, we can see that the variance $Var[\hat{I}] = \infty$ if $k = 2$.

2. Theoretical study

2.1.2.2. Three-dimensional

The corrected irradiance estimate is:

$$\widehat{PD}(r_k) = \frac{(k-1)\phi}{\frac{4}{3}\pi r_k^3} = \frac{(k-1)PD}{nr_k^3} \quad (2.11)$$

and the expected value is [GHUS12]:

$$E[\widehat{PD}(r_k)] = PD \quad (2.12)$$

The variance value is:

$$\begin{aligned} Var[\widehat{PD}(r_k)] &= \int_0^1 \widehat{PD}^2(r_k) f_{X(k)}(r_k) dr_k - PD^2 \\ &= PD^2 \frac{n-k+1}{n(k-2)} \end{aligned} \quad (2.13)$$

From formula 2.13, we can see that the variance $Var[\hat{I}] = \infty$ if $k = 2$.

2.2. Epanechnikov kernel

For epanechnikov kernel we offer here the formula of variance in both 2D and 3D cases.

Two-dimensional

The epanechnikov kernel was used by Walter [Wal98].

$$K_{r_k}(r_i) = \frac{2}{\pi r_k^2} \left[1 - \left(\frac{r_i}{r_k} \right)^2 \right] \quad (2.14)$$

The estimator is:

$$\hat{I}(r_1, \dots, r_k) = \sum_{i=1}^k K_{r_k}(r_i) \phi \quad (2.15)$$

The expected value is [GHUPS14]:

$$E[\hat{I}] = I(P) \quad (2.16)$$

By using the formula 1.12, we can calculate the variance value of the epanechnikov kernel.

The variance value is:

$$Var[\hat{I}] = I^2(P) \left(\frac{4n-3k+2}{3n(k-2)} \right) \quad (2.17)$$

From the formula 2.17, we can see that the variance $Var[\hat{I}] = \infty$ if $k = 2$.

Three-dimensional

The epanechnikov kernel in 3D case is.

$$K_{r_k}(r_i) = \frac{15}{8\pi r_k^3} \left[1 - \left(\frac{r_i}{r_k} \right)^2 \right] \quad (2.18)$$

The estimator is:

$$\hat{I}(r_1, \dots, r_k) = \sum_{i=1}^k K_{r_k}(r_i) \phi \quad (2.19)$$

The expected value is:

$$\begin{aligned} E[\hat{I}] &= \int_0^1 \int_0^{r_k} \hat{I}_i^*(r_i, r_k) f_{X_{(i,k)}}(r_i, r_k) dr_i dr_k \\ &= \frac{3n\phi}{4\pi} = PD \end{aligned} \quad (2.20)$$

By using the formula 1.12, we can calculate the variance value of the epanechnikov kernel.

The variance value is:

$$Var[\hat{I}] = \frac{1}{28} PD^2 \left(\frac{7k(9n-4) - 86n + 16}{(k-2)n} + \frac{35[\Gamma(k-1)]^2}{[\Gamma(k+\frac{2}{3})]^2} (u(k) + v(k)) \right) \quad (2.21)$$

where function $u(n)$ and $v(n)$ with $n \in \mathbb{N}$ defined as:

$$\begin{aligned} u(1) &= 0 & u(2) &= \left[\Gamma\left(\frac{5}{3}\right) \right]^2 \\ u(n) &= \frac{(18n^2 - 60n + 52)u(n-1) - (4-3n)^2 u(n-2)}{9(n-2)^2} \end{aligned} \quad (2.22)$$

and

$$\begin{aligned} v(1) &= 0 & v(2) &= \left[\Gamma\left(\frac{5}{3}\right) \right]^2 \\ v(n) &= \frac{(18n^3 - 87n^2 + 148n - 88)v(n-1) - (4-3n)^2 (n-1)v(n-2)}{9(n-2)^3} \end{aligned} \quad (2.23)$$

The same as the 2D case, from the formula 2.21, we can see that the variance $Var[\hat{I}] = \infty$ if $k = 2$.

2.3. Silverman/Quartic kernel

The silverman/quartic kernel was used by Shirley [WMHSPG98].

$$K_{r_k}(r_i) = \frac{3}{\pi r_k^2} \left[1 - \left(\frac{r_i}{r_k} \right)^2 \right]^2 \quad (2.24)$$

2. Theoretical study

The estimator, as with previous sections, is.

$$\hat{I}(r_1, \dots, r_k) = \sum_{i=1}^k K_{r_k}(r_i) \phi \quad (2.25)$$

The expected value is [GHUPS14]:

$$E[\hat{I}] = I(P) \quad (2.26)$$

By using the formula 1.12, we can calculate the variance value of the silverman/quartic kernel.

The variance value is:

$$Var[\hat{I}] = I^2(P) \left(\frac{9n - 5k + 1}{5n(k - 2)} \right) \quad (2.27)$$

Also here from the formula 2.17, we can see that the variance $Var[\hat{I}] = \infty$ if $k = 2$.

2.4. Cone filter

The cone filter was used by Jensen [Jen96a]. Let us call s the slope of the cone filter.

$$K_{r_k}(r_i) = \frac{1 - \frac{r_i}{sr_k}}{\pi r_k^2 \left(1 - \frac{2}{3s}\right)} \quad (2.28)$$

The estimator is:

$$\hat{I}(r_1, \dots, r_k) = \sum_{i=1}^k K_{r_k}(r_i) \phi \quad (2.29)$$

We will use the formula 1.12 to calculate the variance value. For all the following formulas, $k \geq 3$.

2.4.1. Triangle kernel

If $s = 1$, we have the triangle kernel, and from García et al.[GHUPS14]:

$$E[\hat{I}] = I(P) \quad (2.30)$$

The variance value of triangle kernel is:

$$Var[\hat{I}] = I^2(P) \left(4 + \frac{3n - 2k + 1}{2n(k - 2)} + \frac{3[\Gamma(k - 1)]^2}{[\Gamma(k + \frac{1}{2})]^2} (u(k) - 4v(k)) \right) \quad (2.31)$$

where function $u(k)$ and $v(k)$ with $k \in \mathbb{N}$ defined as:

$$\begin{aligned} u(1) &= 0 & u(2) &= \frac{\pi}{4} \\ u(k) &= \frac{(8k^2 - 28k + 25)u(k-1) - (2k-3)^2 k(k-2)}{4(k-2)^2} \end{aligned} \quad (2.32)$$

and

$$\begin{aligned} v(1) &= 0 & v(2) &= \frac{\pi}{4} \\ v(k) &= \frac{(8k^3 - 40k^2 + 69k - 41)v(k-1) - (k-1)(2k-3)^2 v(k-2)}{4(k-2)^3} \end{aligned} \quad (2.33)$$

2.4.2. Corrected photon mapping with cone filter

The expected value of corrected photon mapping is [GHUPS14]:

$$E[\hat{I}] = I(P) \quad (2.34)$$

The variance value is:

$$Var_{k-1} = I^2(P) \left(\frac{n-k+1}{n(k-2)} + \frac{1}{(2-3s)^2} \left[4 + \frac{n-1}{2n(k-2)} + \frac{3[\Gamma(k-1)]^2}{[\Gamma(k+\frac{1}{2})]^2} (u(k) - 4v(k)) \right] \right) \quad (2.35)$$

where function $u(k)$ and $v(k)$ with $k \in \mathbb{N}$ defined in 2.32 and 2.33

2.4.3. Original photon mapping with cone filter

The expected value of original photon mapping is [GHUPS14]:

$$\begin{aligned} E[\hat{I}] &= E_{k-1} + E[\hat{I}_k^*] \\ &= I(P) + \frac{(3s-3)I(P)}{(3s-2)(k-1)} \end{aligned} \quad (2.36)$$

The variance value of the k^{th} photon is:

$$Var[I_k] = I^2(P) \frac{3(n-k+1)((6k-3)s^2 - (10k-4)s + 4k-1)}{(k-2)(k-1)^2 n(2-3s)^2} \quad (2.37)$$

2. Theoretical study

The variance value is then:

$$\begin{aligned}
Var[\hat{I}] &= Var_{k-1} + Var[\hat{I}_k] \\
&= I^2(P) \left(\frac{n-k+1}{n(k-2)} + \frac{1}{(2-3s)^2} \left[4 + \frac{n-1}{2n(k-2)} + \frac{3[\Gamma(k-1)]^2}{[\Gamma(k+\frac{1}{2})]^2} (u(k) - 4v(k)) \right] \right) \\
&\quad + I^2(P) \frac{3(n-k+1) \left((6k-3)s^2 - (10k-4)s + 4k-1 \right)}{(k-2)(k-1)^2 n(2-3s)^2}
\end{aligned} \tag{2.38}$$

where function $u(k)$ and $v(k)$ with $k \in \mathbb{N}$ defined in 2.32 and 2.33

2.5. Gaussian filter

The gaussian filter described in Jensen's PhD [Jen96b] is:

$$K_{r_k}(r_i) = \frac{1}{\pi r_k^2} \alpha \left[1 - \frac{1 - e^{-\beta \frac{r_i^2}{2r_k^2}}}{1 - e^{-\beta}} \right] \tag{2.39}$$

where $\alpha = 0.918$ and $\beta = 1.953$.

There are four variants of this algorithm: the contribution of the k^{th} nearest photon may or may not be taken into account, and there are two possible values of α . The corrected algorithm with $\tilde{\alpha} = 1.728$ and $\beta = 1.953$ and discarding photon k has an expected value [GHUPS14] of:

$$E_{k-1} = I(P) \tag{2.40}$$

while the other versions are biased.

The variance value is:

$$Var[\hat{I}] = \sum_{i=1}^{k-1} Var[\hat{I}_i^*] + 2 \sum_{1 \leq i < j \leq k-1} Cov[\hat{I}_i^*, \hat{I}_j^*] \tag{2.41}$$

However, to the best of the authors knowledge, for the sum of the upper triangular elements of covariance matrix, there is no analytical formula, so there is no closed analytical formula for gaussian filter, we will show here the formula of the sum of the diagonal of covariance matrix $\sum_{i=1}^{k-1} Var[\hat{I}_i^*]$ in the appendix. Here we offer the preliminary work in the hope that sparks interest in the mathematics community.

$$\begin{aligned}
E[\hat{I}_i^* \hat{I}_j^*] &= 8 \left(\frac{\phi \alpha}{\pi} \right)^2 \mathcal{C}_{\textcircled{1}} \int_0^1 \frac{(1-r_k^2)^{n-k}}{r_k^3} \int_0^{r_k} \left(1 - \frac{1 - e^{-\beta \frac{r_j^2}{2r_k^2}}}{1 - e^{-\beta}} \right) r_j (r_k^2 - r_j^2)^{k-j-1} \times \\
&\quad \int_0^{r_j} \left(1 - \frac{1 - e^{-\beta \frac{r_i^2}{2r_k^2}}}{1 - e^{-\beta}} \right) r_i^{2i-1} (r_j^2 - r_i^2)^{j-i-1} dr_i dr_j dr_k
\end{aligned} \tag{2.42}$$

$$\begin{aligned}
 \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} E[\hat{I}_i^*] E[\hat{I}_j^*] &= \left(\frac{2\phi\alpha\mathcal{C}_J}{\pi} \right)^2 \times \\
 &\left(\int_0^1 \int_0^{r_{k_1}} \frac{(1-r_{k_1}^2)^{n-k}}{r_{k_1}^3} \left(1 - \frac{1-e^{-\beta\frac{r_j^2}{2r_{k_1}^2}}}{1-e^{-\beta}} \right) r_j^{2j-1} (r_{k_1}^2 - r_j^2)^{k-j-1} dr_j dr_{k_1} \right) \times \\
 &\left(\int_0^1 \int_0^{r_{k_2}} \frac{(1-r_{k_2}^2)^{n-k}}{r_{k_2}^3} \left(1 - \frac{1-e^{-\beta\frac{r_i^2}{2r_{k_2}^2}}}{1-e^{-\beta}} \right) r_i^{2i-1} (r_{k_2}^2 - r_i^2)^{k-i-1} dr_i dr_{k_2} \right)
 \end{aligned} \tag{2.43}$$

2.6. Specialized differential filter

The differential filter [JC95] detects discontinuities in radiance by noticing that the changes in the estimate become monotonic near the edges. In this case, the algorithm will stop adding photons and use an estimate with fewer photons [GHUPS14]. So we calculate the variance value with $\tilde{k} \leq k$.

We assume that, the number of photons n is infinity. So for corrected photon mapping with constant kernel we have:

$$\lim_{n \rightarrow \infty} \frac{Var[\hat{I}, k]}{Var[\hat{I}, \tilde{k}]} = \frac{\frac{(n-k+1)I^2(P)}{(k-2)n}}{\frac{(n-\tilde{k}+1)I^2(P)}{(\tilde{k}-2)n}} = \frac{\tilde{k}-2}{k-2} \leq 1 \tag{2.44}$$

For other filters we get similar results, and for all filters, that we have studied, we have

$$Var[\hat{I}, k] \leq Var[\hat{I}, \tilde{k}] \tag{2.45}$$

for $k \geq \tilde{k}$.

So we can see, when k is smaller, the Variance $Var[\hat{I}]$ is greater. Meanwhile, the bias is smaller.

3. Empirical study

In this chapter we will introduce our empirical study. To extend the framework in García et al. [GHUS12] and [GHUPS14], we have implemented in C++ the same scene that presented in the papers above, which consisting of a planar unit disc illuminated by a directional light source, that can create the constant value of the incoming flux, in the case, we have uniform irradiance $I(P)$, and all photons carry the same flux. Then we have calculated the photon mapping signal-to-noise ratio for $n = 100\,000$ and k ranging between 3 and 200, by using constant kernel, the epanechnikov kernel, the silverman/quartic kernel and the cone filter inclusive triangle kernel. We have repeated the simulations ten thousand times to calculate the signal-to-noise ratio of the convergence of the algorithm. After it, we have calculated the signal-to-noise ratio for different kernels under the same condition by using the analytical formulas from our theoretical study. From the both results, we have calculated the square residuals to measure the goodness of fit. In addition, we have compared the calculation time as well. For the signal-to-noise ratio calculation, instead of using the correct expected value, many people use just $I(P)$ as the expected value, we call the result in this case in our thesis the pseudo signal to noise ratio. Besides the signal-to-noise ratio, we have calculated the pseudo signal to noise ratio, then compare the result to the normal signal-to-noise ratio.

3.1. Signal to noise ratio

The signal-to-noise ratio is a measure used in science and engineering to quantify how much a signal has been corrupted by noise. The rose criterion [Ros73] states that an SNR of at least 5 is needed to be able to distinguish image features at 100% certainty. From the signal to noise graphs that we have offered in our thesis, we could find the minimum value of k of different kernels for our needs. In addition, the signal to noise ratio value comparison among different filters allow us to choose the best kernel for our needs. The theoretical prediction of the signal to noise ratio is calculated by $SNR = \frac{\mu}{\sigma}$, where μ is the expected value from García et al. [GHUS12] and [GHUPS14] and σ is the standard deviation, that equals to square root of our variance value, which is a function of k and n . For cone filter, our function is not continuous, but we are interpolating the values to make the graph easier to read.

We get our theoretical results for $n = 100\,000$ and k ranging between 3 and 200 using different kernels within 0.1 sec in average. Regarding to the calculation cost, for our empirical calculation, it takes 5 hours for constant kernel and for other kernels, it takes even longer. So it is efficient to use our formula to calculate the variance value.

Figures 3.1 and 3.2 show the signal to noise Ratio of different filters of the photon mapping. In figure 3.1: The first row show the cases of cone filter using $s=0.5$ for original photon mapping and corrected photon mapping. The second row show the cases of cone filter using $s=0.75$. The bottom row show the case of cone filter using $s=1.5$. In figure 3.2: The first row shows the cases of constant kernel of original photon mapping and corrected photon mapping. The second row and the third row show the cases of epanechnikov kernel,

3. Empirical study

silverman/quartic kernel and triangle kernel of corrected photon mapping. The theoretical results are represented by green line, The empirical results are represented by red points. We can see in the pictures almost only the green line with a couple of red points behind it, that means, our theoretical result matches well with the empirical results. Meanwhile, we offer here an overview for all kernels above, we can see that the original photon mapping with constant kernel and the corrected photon mapping with constant kernel have the same signal to noise ratio, and the highest signal to noise ratio as well. That is because the calculation without filters is simpler, but if we want a much smoother result, then it is better to use the filters for improving radiance estimation. Follows constant kernel, we can see the cone filter with $s=1.5$ and epanichnikov filter. The signal to noise ratio of cone filter with $s=1.5$ is slightly higher, but from our study, epanichnikov filter takes less calculation time than cone filter. Moreover, from García et al. [GHUPS14], we already know that the estimator provided by the cone filter is not consistent unless the slope is one, and that the epanechnikov and silverman kernels are consistent. The triangle filter and silverman filter follow directly the epanichnikov filter. Then follows the cone filter with $s=0.5$ and $s=0.75$. The same as cone filter with $s=1.5$, we can see the underestimation bias in original photon mapping with cone filters.

3.2. Pseudo signal to noise ratio

For original photon mapping, which the contribution of impact k is counted, the expected value of constant kernel is $\frac{k}{k-1}I(P)$ and the expected value of cone filter is $I(P) + \frac{(3s-3)I(P)}{(3s-2)(k-1)}$, But instead of the correct formulas, many people use just $I(P)$ as the expected value. We can also model this typical implementation mistakes and quantify their effect. Here we will name $\frac{I(P)}{\sigma}$ as ‘‘Pseudo signal to noise ratio’’, and denote it \widetilde{SNR} .

Figure 3.3 shows the pseudo signal to noise ratio \widetilde{SNR} of different filters of the original photon mapping. The first row shows the cases of constant kernel and cone filter with $s=0.5$, the second row shows the the cases of cone filter with $s=0.75$ and $s=1.5$. The predicted value is represented by green line and the empirical results are represented by red points. As we can see, positively biased algorithms can have misleadingly high signal to noise ratios. To avoid the issue, García et al. [GHUPS14] used in figure 5 the unbiased irradiance $I(P)$ instead of the biased estimator $E[I(P)]$ for the numerator of $\frac{\mu}{\sigma}$.

Figure 3.4 shows the theoretical signal to noise ratio compares to the pseudo signal to noise ratio \widetilde{SNR} . The first row shows the cases of constant kernel and cone filter with $s=0.5$, the second row shows the the cases of cone filter with $s=0.75$ and $s=1.5$. The theoretical signal to noise ratio is represented by green line, the pseudo signal to noise ratio is represented by red line. From the graphs we can see the overestimation and underestimation for each case. For cone filter with $s=0.5$ and $s=0.75$ the overestimation and underestimation biases are significant.

3.2. Pseudo signal to noise ratio

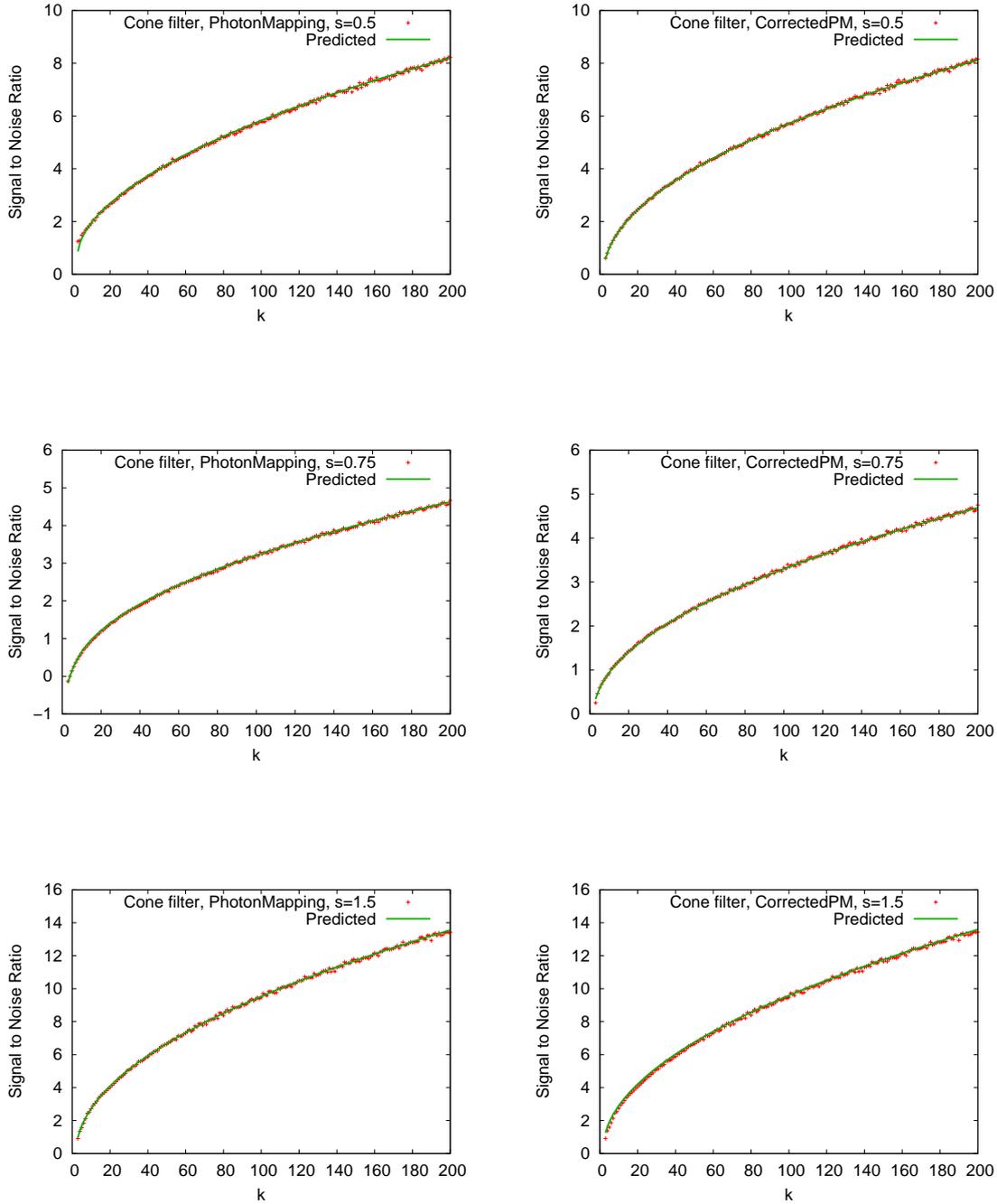


Figure 3.1.: Signal to noise ratio of the cone filter of photon mapping. First row: Original photon mapping with cone filter using $s=0.5$ (left) and corrected photon mapping with cone filter using $s=0.5$ (right). Second row: Original photon mapping with cone filter using $s=0.75$ (left) and corrected photon mapping with cone filter using $s=0.75$ (right). Third row: Original photon mapping with cone filter using $s=1.5$ (left) and corrected photon mapping with cone filter using $s=1.5$ (right). Uniform distribution of photons as a function of k (in red); theoretical prediction of the signal to noise ratio (in green).

3. Empirical study

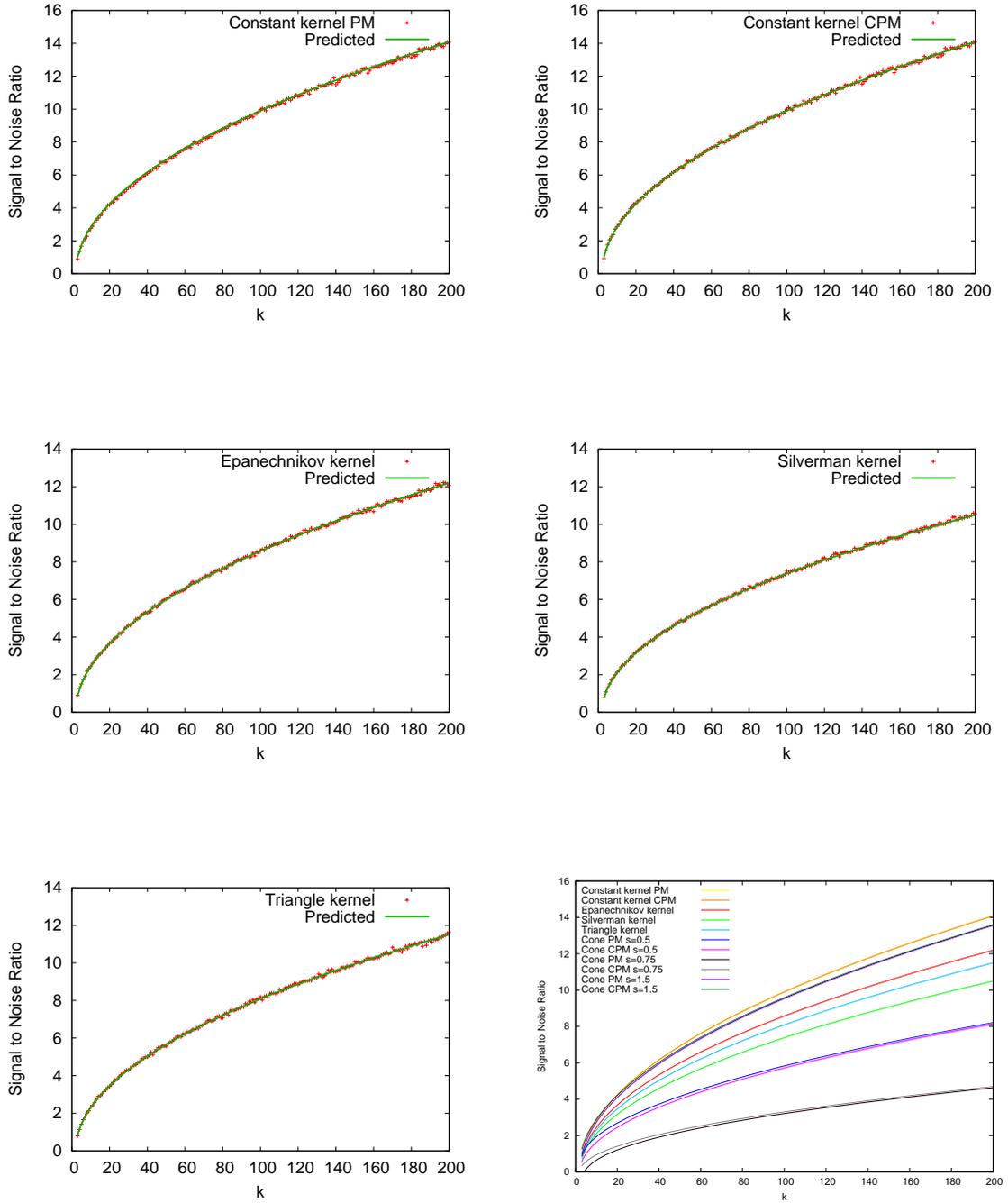


Figure 3.2.: Signal to noise ratio of the different filters of photon mapping. First row: Original photon mapping with constant kernel (left), corrected photon mapping with constant kernel (right). Second row: Corrected photon mapping with epanechnikov filter (left) and silverman/quartic filter (right). Third row: Corrected photon mapping with triangle filter (left) and an overview for all kernels above (right). Uniform distribution of photons as a function of k (in red); theoretical prediction of the signal to noise ratio (in green).

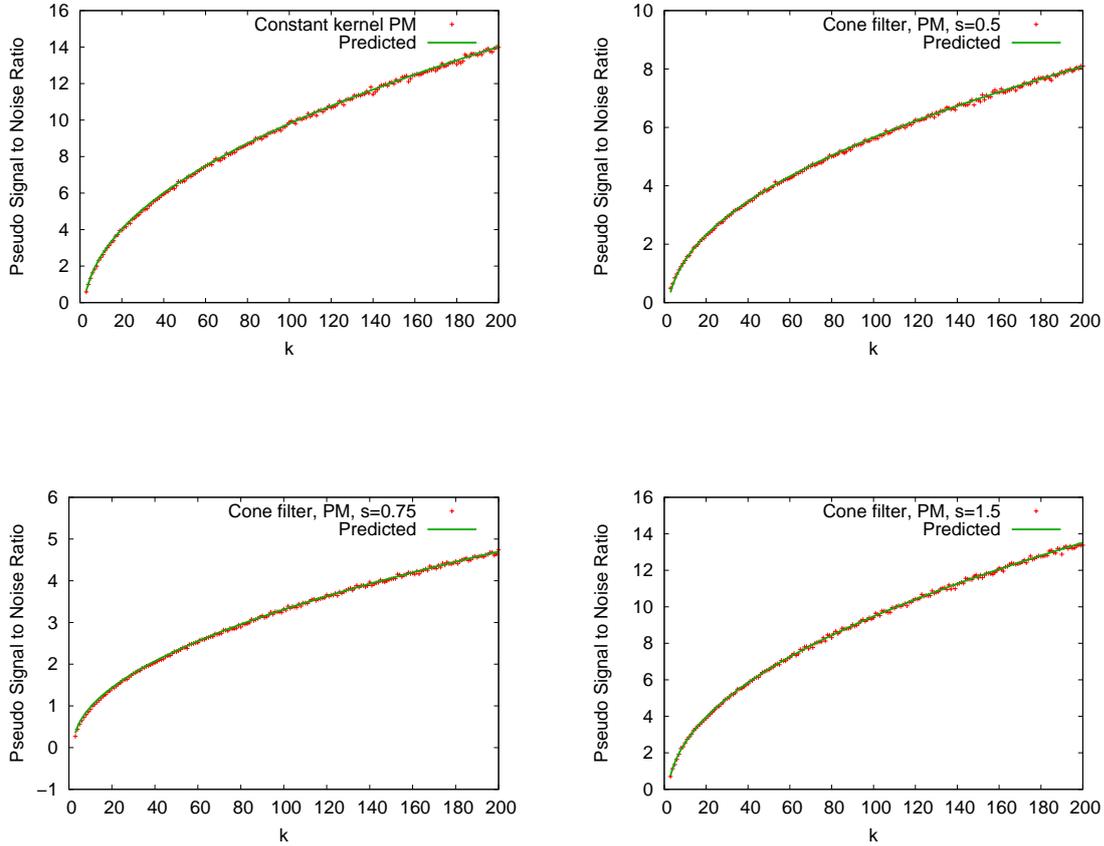


Figure 3.3.: Pseudo signal to noise ratio of the different filters of original photon mapping. First row: original photon mapping with constant kernel (left) and cone filter using $s=0.5$ (right). Second row: original photon mapping with cone filter using $s=0.75$ (left) and $s=1.5$ (right). Uniform distribution of photons as a function of k (in red); theoretical prediction of the signal to noise ratio (in green).

3. Empirical study

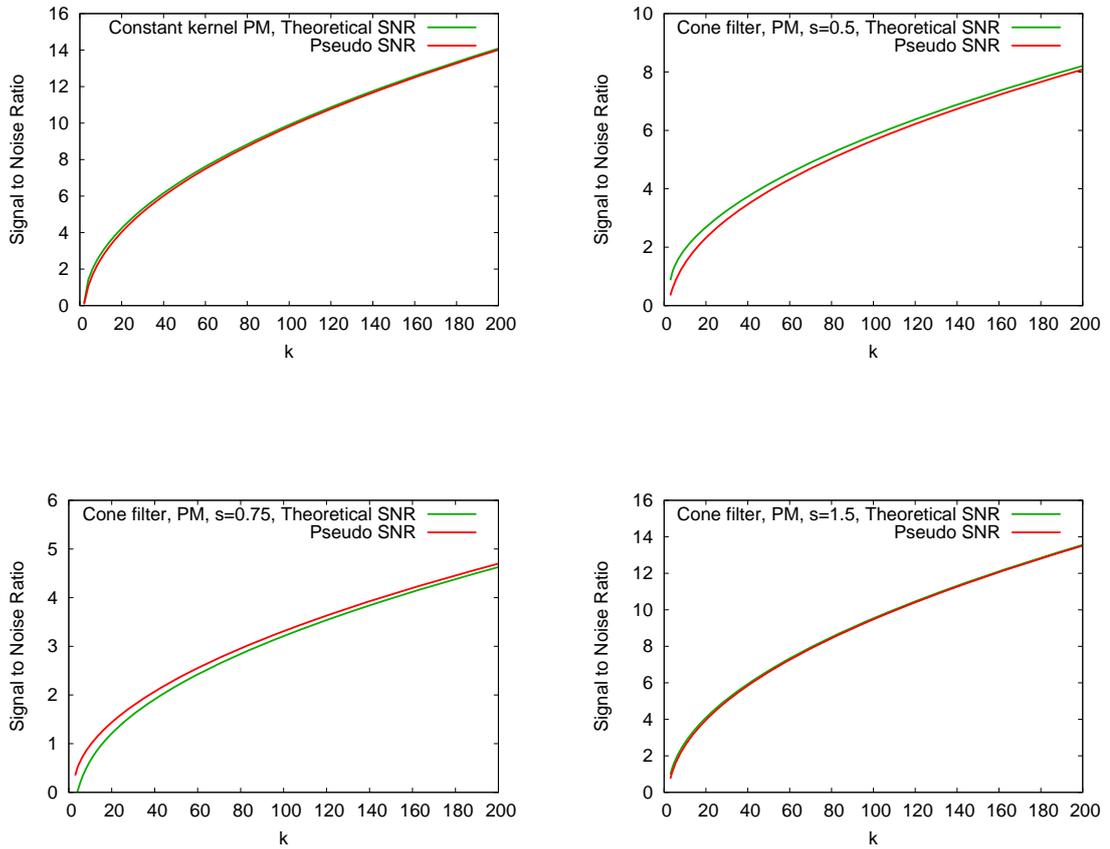


Figure 3.4.: Theoretical signal to noise ratio versus pseudo signal to noise ratio for the different filters of original photon mapping. First row: original photon mapping with constant kernel (left) and cone filter using $s=0.5$ (right). Second row: original photon mapping with cone filter using $s=0.75$ (left) and $s=1.5$ (right). Theoretical signal to noise ratio (in green); Pseudo signal to noise ratio (in red)

4. Conclusions and Future Work

In this chapter, we give here our conclusions and our plan for the future work.

4.1. Conclusions

We have extended the study that presented in García et al. [GHUPS14] to determine the variance of photon mapping with different filters, and provided a theoretical and empirical study of the filtering kernels in the context of photon mapping.

From our theoretical study, we have got the analytical formulas of the variance value for constant kernel and epanechnikov filter in both 2D and 3D case as well as silverman/quartic kernel, triangle kernel in the context of photon mapping, they are all functions of k , the number of photons sought in a photon maps query and n , the number of photon impacts. We have got the analytical formula of the variance value of the cone filter, which is a function of n , k , and s , the slope of the cone filter. We have showed that the computation time of our formulas is very low, we could get the signal to noise ratio for different kernels within 0.1 sec in average. But actually, the print function in C++ takes a lot of time, so our true computation cost is much less than 0.1 sec. Moreover, we have also studied the differential filter, and we have found out the variance becomes smaller when k is greater, so the bias is then smaller.

During our study we realized that the value of k must be greater than 2 for all the kernels, that we have studied, otherwise the variance value is infinity if $k = 2$, because the denominator in the formulas would be 0. For example, for corrected photon mapping with constant kernel, the variance $Var[\hat{I}] = I^2(P) \frac{n-k+1}{n(k-2)}$, it is obviously that $n(k-2) = 0$, if $k = 2$, the variance value is then infinity. For the case when $k = 1$, the variance value is either infinity or negative. As we can easily see from our analytical formulas, for constant kernel, if $k = 1$, the variance value of the corrected photon mapping $Var[\hat{I}] = I^2(P) \frac{n-k+1}{n(k-2)} = -I^2(P)$; the variance value of the original photon mapping is $Var[\hat{I}] = \frac{k^2(n-k+1)I^2(P)}{(k-1)^2(k-2)n} = Infinity$, because the denominator $(k-1)^2(k-2)n = 0$.

In our empirical study, we have implemented a scene consisting of a planar unit disc, a very simple model, which is illuminated by a directional light source for each kernel and calculated the signal to noise ratio of each kernel. We have calculated the theoretical prediction of the signal to noise ratio as well. Moreover, we show the both results in a graph for every kernel, that we have studied. We have calculated the squared residuals between the theoretical result and the empirical result to measure the goodness of fit [LLJ16]. The squared residuals are less than 3500 ppm for all different filters. It shows, that our theoretical result and the empirical result match well.

For constant kernel, the original photon mapping and the corrected photon mapping from García et al. [GHUS12] have the same signal-to-noise ratio. The theoretical comparison among different filters allows us to choose the best kernel for our needs. Besides, we could

4. Conclusions and Future Work

offer a threshold, the estimation of computational cost of algorithms which stop after error is smaller than the threshold.

4.2. Future Works

Besides uniform distribution, we also want to simulate a scene composed of a disc illuminated by a point light source located above the disc centre at one unit distance.

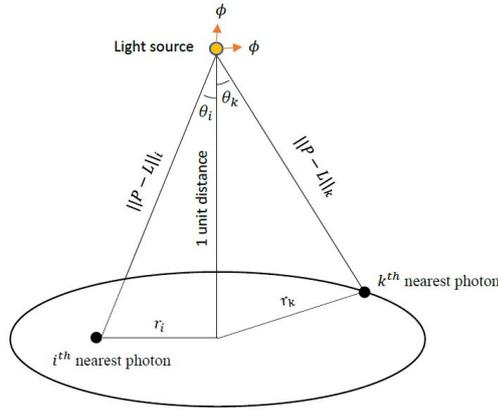


Figure 4.1.: The point light source model

The fraction of the power which leaves the light source and arrives at the point P in the plane (the form factor) is $\frac{\cos(\theta)}{\|P-L\|^2}$, with θ denoting the angle between the light vector and the normal, L the light position and P the point of interest.

Figure 4.1 shows our model with point light source. Let ϕ be the power which leaves the light source, then the power of the k^{th} nearest photon is:

$$\widehat{I}_k^* = \frac{\cos(\theta)}{\|P-L\|^2} \phi = \frac{1}{\|P-L\|^3} \phi = (r_k^2 + 1)^{-\frac{3}{2}} \phi \quad (4.1)$$

Our near term future work will deal with the Variance of different kernels by point light source.

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A. Appendix

A.1. Prerequisites

Because for the order statistics $i < j < k$, we get:

$$\begin{aligned}
 Var[\hat{I}] &= \sum_{i=1}^{k-1} Var[\hat{I}_i^*] + 2 \sum_{1 \leq i < j \leq k-1} (E[\hat{I}_i^* \hat{I}_j^*] - E[\hat{I}_i^*]E[\hat{I}_j^*]) \\
 &= \sum_{i=1}^{k-1} Var[\hat{I}_i^*] + 2 \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} E[\hat{I}_i^* \hat{I}_j^*] - 2 \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} E[\hat{I}_i^*]E[\hat{I}_j^*]
 \end{aligned} \tag{A.1}$$

A.2. Original PM with constant kernel

Two-dimensional

The variance is:

$$Var[\hat{I}] = \int_0^1 \hat{I}^2(r_k) f_{X_{(k)}}(r_k) dr_k - \frac{k^2}{(k-1)^2} I^2(P) \tag{A.2}$$

A.2.0.1. Integral

$$\begin{aligned}
 \int_0^1 \hat{I}^2(r_k) f_{X_{(k)}}(r_k) dr_k &= \int_0^1 \left(\frac{k\phi}{\pi r_k^2} \right)^2 \mathcal{C} F(r_k)^{k-1} (1 - F(r_k))^{n-k} f(r_k) dr_k \\
 &= \frac{k^2 \phi^2}{\pi^2} \mathcal{C} \int_0^1 \frac{1}{r_k^4} F(r_k)^{k-1} (1 - F(r_k))^{n-k} f(r_k) dr_k \\
 &= \frac{k^2 \phi^2}{\pi^2} \mathcal{C} \int_0^1 \frac{1}{F(r_k)^2} F(r_k)^{k-1} (1 - F(r_k))^{n-k} f(r_k) dr_k \\
 &= \frac{k^2 \phi^2}{\pi^2} \mathcal{C} \int_0^1 F(r_k)^{k-3} (1 - F(r_k))^{n-k} f(r_k) dr_k \\
 &= \frac{k^2 \phi^2}{\pi^2} \mathcal{C} \frac{\Gamma(k-2)\Gamma(n-k+1)}{\Gamma(n-1)} \\
 &= \frac{k^2 \phi^2}{\pi^2} \frac{n!}{(k-1)!(n-k)!} \frac{(k-3)!(n-k)!}{(n-2)!} \\
 &= \frac{k^2 \phi^2}{\pi^2} \frac{n(n-1)}{(k-1)(k-2)} \\
 &= \frac{k^2 \pi^2 I^2(P)}{\pi^2 n^2} \frac{n(n-1)}{(k-1)(k-2)} \\
 &= \frac{k^2 I^2(P)}{n} \frac{(n-1)}{(k-1)(k-2)}
 \end{aligned} \tag{A.3}$$

A.2.0.2. Variance

$$\begin{aligned}
 Var[\hat{I}] &= \frac{k^2 I^2(P)}{n} \frac{(n-1)}{(k-1)(k-2)} - \frac{k^2}{(k-1)^2} I^2(P) \\
 &= \frac{k^2(n-1)(k-1) - k^2 n(k-2)}{n(k-1)^2(k-2)} I^2(P) \\
 &= \frac{k^2(n-k+1)I^2(P)}{(k-1)^2(k-2)n}
 \end{aligned} \tag{A.4}$$

Three-dimensional

The variance is:

$$Var[\widehat{PD}(r_k)] = \int_0^1 \widehat{PD}^2(r_k) f_{X(k)}(r_k) dr_k - \frac{k^2}{(k-1)^2} PD^2 \tag{A.5}$$

A.2.0.3. Integral

$$\begin{aligned}
 \int_0^1 \widehat{PD}^2(r_k) f_{X(k)}(r_k) dr_k &= \int_0^1 \left(\frac{k\phi}{\frac{4}{3}\pi r_k^3} \right)^2 \mathcal{C} F(r_k)^{k-1} (1-F(r_k))^{n-k} f(r_k) dr_k \\
 &= \frac{9k^2\phi^2}{16\pi^2} \mathcal{C} \int_0^1 \frac{1}{r_k^6} F(r_k)^{k-1} (1-F(r_k))^{n-k} f(r_k) dr_k \\
 &= \frac{9k^2\phi^2}{16\pi^2} \mathcal{C} \int_0^1 \frac{1}{F(r_k)^2} F(r_k)^{k-1} (1-F(r_k))^{n-k} f(r_k) dr_k \\
 &= \frac{9k^2\phi^2}{16\pi^2} \mathcal{C} \int_0^1 F(r_k)^{k-3} (1-F(r_k))^{n-k} f(r_k) dr_k \\
 &= \frac{9k^2\phi^2}{16\pi^2} \mathcal{C} \int_0^1 (r_k)^{3(k-3)} (1-(r_k)^3)^{n-k} \mathfrak{z}(r_k)^2 dr_k \\
 &= \frac{9k^2\phi^2}{16\pi^2} \mathcal{C} \frac{\Gamma(k-2)\Gamma(n-k+1)}{\Gamma(n-1)} \\
 &= \frac{9k^2\phi^2}{16\pi^2} \frac{n!}{(k-1)!(n-k)!} \frac{(k-3)!(n-k)!}{(n-2)!} \\
 &= \frac{9k^2\phi^2}{16\pi^2} \frac{n(n-1)}{(k-1)(k-2)} \\
 &= \frac{9k^2}{16\pi^2} \frac{16\pi^2 PD^2}{9n^2} \frac{n(n-1)}{(k-1)(k-2)} \\
 &= \frac{k^2 PD^2}{n} \frac{(n-1)}{(k-1)(k-2)}
 \end{aligned} \tag{A.6}$$

A.2.0.4. Variance

$$\begin{aligned}
\text{Var}[\widehat{PD}(r_k)] &= \frac{k^2 PD^2}{n} \frac{(n-1)}{(k-1)(k-2)} - \frac{k^2}{(k-1)^2} PD^2 \\
&= \frac{k^2(n-1)(k-1) - k^2 n(k-2)}{n(k-1)^2(k-2)} PD^2 \\
&= \frac{k^2(n-k+1)PD^2}{(k-1)^2(k-2)n}
\end{aligned} \tag{A.7}$$

A.3. Corrected PM with constant kernel

Two-dimensional

The variance is:

$$\text{Var}[\hat{I}] = \int_0^1 \hat{I}^2(r_k) f_{X_{(k)}}(r_k) dr_k - I^2(P) \tag{A.8}$$

A.3.0.1. Integral

$$\begin{aligned}
\int_0^1 \hat{I}^2(r_k) f_{X_{(k)}}(r_k) dr_k &= \int_0^1 \left(\frac{(k-1)\phi}{\pi r_k^2} \right)^2 \mathcal{C} F(r_k)^{k-1} (1-F(r_k))^{n-k} f(r_k) dr_k \\
&= \frac{(k-1)^2 \phi^2}{\pi^2} \mathcal{C} \int_0^1 \frac{1}{r_k^4} F(r_k)^{k-1} (1-F(r_k))^{n-k} f(r_k) dr_k \\
&= \frac{(k-1)^2 \phi^2}{\pi^2} \mathcal{C} \int_0^1 \frac{1}{F(r_k)^2} F(r_k)^{k-1} (1-F(r_k))^{n-k} f(r_k) dr_k \\
&= \frac{(k-1)^2 \phi^2}{\pi^2} \mathcal{C} \int_0^1 F(r_k)^{k-3} (1-F(r_k))^{n-k} f(r_k) dr_k \\
&= \frac{(k-1)^2 \phi^2}{\pi^2} \mathcal{C} \frac{\Gamma(k-2)\Gamma(n-k+1)}{\Gamma(n-1)} \\
&= \frac{(k-1)^2 \phi^2}{\pi^2} \frac{n!}{(k-1)!(n-k)!} \frac{(k-3)!(n-k)!}{(n-2)!} \\
&= \frac{(k-1)^2 \phi^2}{\pi^2} \frac{n(n-1)}{(k-1)(k-2)} \\
&= \frac{(k-1)^2 \pi^2 I^2(P)}{\pi^2} \frac{n(n-1)}{n^2 (k-1)(k-2)} \\
&= \frac{(k-1)I^2(P)}{n} \frac{(n-1)}{(k-2)}
\end{aligned} \tag{A.9}$$

A.3.0.2. Variance

$$\begin{aligned}
 \text{Var}[\hat{I}] &= \frac{I^2(P)}{n} \frac{(n-1)(k-1)}{(k-2)} - I^2(P) \\
 &= I^2(P) \frac{(n-1)(k-1) - n(k-2)}{n(k-2)} \\
 &= I^2(P) \frac{n-k+1}{n(k-2)}
 \end{aligned} \tag{A.10}$$

Three-dimensional

The variance is:

$$\text{Var}[\widehat{PD}(r_k)] = \int_0^1 \widehat{PD}^2(r_k) f_{X_{(k)}}(r_k) dr_k - PD^2 \tag{A.11}$$

A.3.0.3. Integral

$$\begin{aligned}
 \int_0^1 \widehat{PD}^2(r_k) f_{X_{(k)}}(r_k) dr_k &= \int_0^1 \left(\frac{(k-1)\phi}{\frac{4}{3}\pi r_k^3} \right)^2 \mathcal{C} F(r_k)^{k-1} (1-F(r_k))^{n-k} f(r_k) dr_k \\
 &= \frac{9(k-1)^2 \phi^2}{16\pi^2} \mathcal{C} \int_0^1 \frac{1}{r_k^6} F(r_k)^{k-1} (1-F(r_k))^{n-k} f(r_k) dr_k \\
 &= \frac{9(k-1)^2 \phi^2}{16\pi^2} \mathcal{C} \int_0^1 \frac{1}{F(r_k)^2} F(r_k)^{k-1} (1-F(r_k))^{n-k} f(r_k) dr_k \\
 &= \frac{9(k-1)^2 \phi^2}{16\pi^2} \mathcal{C} \int_0^1 (r_k)^{3(k-3)} (1-(r_k)^3)^{n-k} \mathfrak{z}(r_k)^2 dr_k \\
 &= \frac{9(k-1)^2 \phi^2}{16\pi^2} \mathcal{C} \frac{\Gamma(k-2)\Gamma(n-k+1)}{\Gamma(n-1)} \\
 &= \frac{9(k-1)^2 \phi^2}{16\pi^2} \frac{n!}{(k-1)!(n-k)!} \frac{(k-3)!(n-k)!}{(n-2)!} \\
 &= \frac{9(k-1)^2 \phi^2}{16\pi^2} \frac{n(n-1)}{(k-1)(k-2)} \\
 &= \frac{9(k-1)^2}{16\pi^2} \frac{16\pi^2 PD^2}{9n^2} \frac{n(n-1)}{(k-1)(k-2)} \\
 &= \frac{(k-1)PD^2}{n} \frac{(n-1)}{(k-2)}
 \end{aligned} \tag{A.12}$$

A.3.0.4. Variance

$$\begin{aligned}
 \text{Var}[\widehat{PD}(r_k)] &= \frac{PD^2}{n} \frac{(n-1)(k-1)}{(k-2)} - PD^2 \\
 &= PD^2 \frac{(n-1)(k-1) - n(k-2)}{n(k-2)} \\
 &= PD^2 \frac{n-k+1}{n(k-2)}
 \end{aligned} \tag{A.13}$$

A.4. Epanechnikov kernel

The variance is A.1:

$$\text{Var}[\hat{I}] = \sum_{i=1}^{k-1} \text{Var}[\hat{I}_i^*] + 2 \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} E[\hat{I}_i^* \hat{I}_j^*] - 2 \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} E[\hat{I}_i^*] E[\hat{I}_j^*] \quad (\text{A.14})$$

$$\sum_{i=1}^{k-1} \text{Var}[\hat{I}_i^*]$$

$$\text{Var}[\hat{I}_i^*] = E[\hat{I}_i^{*2}] - (E[\hat{I}_i^*])^2 \quad (\text{A.15})$$

So we have

$$\sum_{i=1}^{k-1} \text{Var}[\hat{I}_i^*] = \sum_{i=1}^{k-1} \left(E[\hat{I}_i^{*2}] - (E[\hat{I}_i^*])^2 \right) \quad (\text{A.16})$$

A.4.0.1. $E[\hat{I}_i^{*2}]$

Since $\forall r_i > r_k, f_{X_{(i,k)}}(r_i, r_k) = 0$, the inner integral upper limit is r_k

$$\begin{aligned} E[\hat{I}_i^{*2}] &= \int_0^1 \int_0^{r_k} \hat{I}_i^{*2}(r_i, r_k) f_{X_{(i,k)}}(r_i, r_k) dr_i dr_k \\ &= \int_0^1 \int_0^{r_k} \left(\frac{2}{\pi r_k^2} \left[1 - \left(\frac{r_i}{r_k} \right)^2 \right] \phi \right)^2 C_K 2r_i^{2i-1} [r_k^2 - r_i^2]^{k-i-1} dr_i dr_k \\ &= \int_0^1 \int_0^{r_k} \left(\frac{2}{\pi r_k^2} \left[1 - \left(\frac{r_i}{r_k} \right)^2 \right] \phi \right)^2 2C_J r_k [1 - r_k^2]^{n-k} 2r_i^{2i-1} [r_k^2 - r_i^2]^{k-i-1} dr_i dr_k \\ &= \left(\frac{2\phi}{\pi} \right)^2 C_J \int_0^1 \int_0^{r_k} \frac{1}{r_k^4} \left[1 - \left(\frac{r_i}{r_k} \right)^2 \right]^2 2r_k [1 - r_k^2]^{n-k} 2r_i^{2i-1} [r_k^2 - r_i^2]^{k-i-1} dr_i dr_k \\ &= \left(\frac{2\phi}{\pi} \right)^2 C_J \int_0^1 \frac{2r_k [1 - r_k^2]^{n-k}}{r_k^4} \int_0^{r_k} \left[1 - \left(\frac{r_i}{r_k} \right)^2 \right]^2 2r_i^{2i-1} [r_k^2 - r_i^2]^{k-i-1} dr_i dr_k \end{aligned} \quad (\text{A.17})$$

The value of the inner integral is:

$$\int_0^{r_k} \left[1 - \left(\frac{r_i}{r_k} \right)^2 \right]^2 2r_i^{2i-1} [r_k^2 - r_i^2]^{k-i-1} dr_i = \frac{r_k^{2(k-1)} \Gamma(i) \Gamma(k-i+2)}{\Gamma(k+2)} \quad (\text{A.18})$$

Then we can solve the outer integral:

$$\int_0^1 \frac{2r_k [1 - r_k^2]^{n-k}}{r_k^4} \frac{r_k^{2(k-1)} \Gamma(i) \Gamma(k-i+2)}{\Gamma(k+2)} dr_k = \frac{\Gamma(i) \Gamma(k-2) \Gamma(k-i+2) \Gamma(n-k+1)}{\Gamma(k+2) \Gamma(n-1)} \quad (\text{A.19})$$

A. Appendix

Finally, we have:

$$\begin{aligned}
E[\hat{I}_i^{*2}] &= \left(\frac{2\phi}{\pi}\right)^2 C_J \frac{\Gamma(i)\Gamma(k-2)\Gamma(k-i+2)\Gamma(n-k+1)}{\Gamma(k+2)\Gamma(n-1)} \\
&= \left(\frac{2\phi}{\pi}\right)^2 \frac{n!}{(i-1)!(k-i-1)!(n-k)!} \frac{\Gamma(i)\Gamma(k-2)\Gamma(k-i+2)\Gamma(n-k+1)}{\Gamma(k+2)\Gamma(n-1)} \\
&= \left(\frac{2\phi}{\pi}\right)^2 \frac{n!}{(i-1)!(k-i-1)!(n-k)!} \frac{(i-1)!(k-3)!(k-i+1)!(n-k)!}{(k+1)!(n-2)!} \\
&= \left(\frac{2\phi}{\pi}\right)^2 \frac{n(n-1)(k-i+1)(k-i)}{(k+1)k(k-1)(k-2)}
\end{aligned} \tag{A.20}$$

A.4.0.2. $\sum_{i=1}^{k-1} \left(E[\hat{I}_i^{*2}] - (E[\hat{I}_i^*])^2 \right)$

From García et al. [GHUPS14], we already know that:

$$E[\hat{I}_i^*] = \frac{2\phi}{\pi} \frac{(k-i)n}{(k-1)k} \tag{A.21}$$

So we have now

$$\begin{aligned}
\sum_{i=1}^{k-1} Var[\hat{I}_i^*] &= \sum_{i=1}^{k-1} \left(E[\hat{I}_i^{*2}] - (E[\hat{I}_i^*])^2 \right) \\
&= \sum_{i=1}^{k-1} \left(\left(\frac{2\phi}{\pi}\right)^2 \frac{n(n-1)(k-i+1)(k-i)}{(k+1)k(k-1)(k-2)} - \left(\frac{2\phi}{\pi} \frac{(k-i)n}{(k-1)k}\right)^2 \right) \\
&= \left(\frac{2\phi}{\pi}\right)^2 \sum_{i=1}^{k-1} \left(\frac{n(n-1)(k-i+1)(k-i)}{(k+1)k(k-1)(k-2)} - \left(\frac{(k-i)n}{(k-1)k}\right)^2 \right) \\
&= \left(\frac{\phi}{\pi}\right)^2 n \left(\frac{2n(3k-2)}{3k(k-1)(k-2)} - \frac{4}{3(k-2)} \right)
\end{aligned} \tag{A.22}$$

$$\sum_{j=1}^{k-1} \sum_{i=1}^{j-1} E[\hat{I}_i^* \hat{I}_j^*]$$

$$\begin{aligned}
E[\hat{I}_i^* \hat{I}_j^*] &= \int_0^1 \int_0^{r_k} \int_0^{r_j} \hat{I}_i^*(r_i, r_k) \hat{I}_j^*(r_j, r_k) f_{X_{(i,j,k)}}(r_i, r_j, r_k) dr_i dr_j dr_k \\
&= \int_0^1 \int_0^{r_k} \int_0^{r_j} \left(\frac{2}{\pi r_k^2} \left[1 - \left(\frac{r_i}{r_k}\right)^2 \right] \phi \right) \left(\frac{2}{\pi r_k^2} \left[1 - \left(\frac{r_j}{r_k}\right)^2 \right] \phi \right) \times \\
&\quad \mathcal{C}_{\textcircled{D}} 2r_i 2r_j 2r_k [r_i^2]^{i-1} [r_j^2 - r_i^2]^{j-i-1} [r_k^2 - r_j^2]^{k-j-1} [1 - r_k^2]^{n-k} dr_i dr_j dr_k \\
&= \left(\frac{2\phi}{\pi}\right)^2 \mathcal{C}_{\textcircled{D}} 2^3 \int_0^1 \int_0^{r_k} \int_0^{r_j} \frac{1}{r_k^4} \left[1 - \left(\frac{r_i}{r_k}\right)^2 \right] \left[1 - \left(\frac{r_j}{r_k}\right)^2 \right] \times \\
&\quad r_i r_j r_k [r_i^2]^{i-1} [r_j^2 - r_i^2]^{j-i-1} [r_k^2 - r_j^2]^{k-j-1} [1 - r_k^2]^{n-k} dr_i dr_j dr_k \\
&= 32 \left(\frac{\phi}{\pi}\right)^2 \mathcal{C}_{\textcircled{D}} \int_0^1 \frac{[1 - r_k^2]^{n-k}}{r_k^3} \int_0^{r_k} \left[1 - \left(\frac{r_j}{r_k}\right)^2 \right] r_j [r_k^2 - r_j^2]^{k-j-1} \\
&\quad \int_0^{r_j} \left[1 - \left(\frac{r_i}{r_k}\right)^2 \right] r_i [r_i^2]^{i-1} [r_j^2 - r_i^2]^{j-i-1} dr_i dr_j dr_k
\end{aligned} \tag{A.23}$$

The value of the inner integral is:

$$\begin{aligned} \int_0^{r_j} \left[1 - \left(\frac{r_i}{r_k}\right)^2\right] r_i [r_i^2]^{i-1} [r_j^2 - r_i^2]^{j-i-1} dr_i &= \int_0^{r_j} \left[1 - \left(\frac{r_i}{r_k}\right)^2\right] [r_i]^{2i-1} [r_j^2 - r_i^2]^{j-i-1} dr_i \\ &= \frac{r_j^{2(j-1)} (r_k^2 j - r_j^2 i) \Gamma[i] \Gamma[j-i]}{2r_k^2 \Gamma[j+1]} \end{aligned} \quad (\text{A.24})$$

Then, the value of the middle integral is:

$$\begin{aligned} \int_0^{r_k} \left[1 - \left(\frac{r_j}{r_k}\right)^2\right] r_j [r_k^2 - r_j^2]^{k-j-1} \frac{r_j^{2(j-1)} (r_k^2 j - r_j^2 i) \Gamma[i] \Gamma[j-i]}{2r_k^2 \Gamma[j+1]} dr_j \\ = \frac{(k-i+1) r_k^{2(k-1)} \Gamma[i] \Gamma[j-i] \Gamma[k-j+1]}{4\Gamma[k+2]} \end{aligned} \quad (\text{A.25})$$

Then we can solve the outer integral:

$$\begin{aligned} \int_0^1 \frac{[1 - r_k^2]^{n-k} (k-i+1) r_k^{2(k-1)} \Gamma[i] \Gamma[j-i] \Gamma[k-j+1]}{r_k^3 4\Gamma[k+2]} dr_k \\ = \frac{(k-i+1) \Gamma[i] \Gamma[j-i] \Gamma[k-2] \Gamma[k-j+1] \Gamma[n-k+1]}{8\Gamma[k+2] \Gamma[n-1]} \end{aligned} \quad (\text{A.26})$$

Then, we have:

$$\begin{aligned} E[\hat{I}_i^* \hat{I}_j^*] &= 32 \left(\frac{\phi}{\pi}\right)^2 \mathcal{C}_{\oplus} \frac{(k-i+1) \Gamma[i] \Gamma[j-i] \Gamma[k-2] \Gamma[k-j+1] \Gamma[n-k+1]}{8\Gamma[k+2] \Gamma[n-1]} \\ &= 32 \left(\frac{\phi}{\pi}\right)^2 \frac{n!}{(i-1)! (j-i-1)! (k-j-1)! (n-k)!} \times \\ &\quad \frac{(k-i+1) \Gamma[i] \Gamma[j-i] \Gamma[k-2] \Gamma[k-j+1] \Gamma[n-k+1]}{8\Gamma[k+2] \Gamma[n-1]} \\ &= 32 \left(\frac{\phi}{\pi}\right)^2 \frac{n!}{(i-1)! (j-i-1)! (k-j-1)! (n-k)!} \times \\ &\quad \frac{(k-i+1) (i-1)! (j-i-1)! (k-3)! (k-j)! (n-k)!}{8(k+1)! (n-2)!} \\ &= 4 \left(\frac{\phi}{\pi}\right)^2 \frac{n(n-1)(k-i+1)(k-j)}{(k+1)k(k-1)(k-2)} \end{aligned} \quad (\text{A.27})$$

Finally, we have:

$$\begin{aligned} \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} E[\hat{I}_i^* \hat{I}_j^*] &= \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} \left(4 \left(\frac{\phi}{\pi}\right)^2 \frac{n(n-1)(k-i+1)(k-j)}{(k+1)k(k-1)(k-2)} \right) \\ &= 4n(n-1) \left(\frac{\phi}{\pi}\right)^2 \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} \left(\frac{(k-i+1)(k-j)}{(k+1)k(k-1)(k-2)} \right) \\ &= \left(\frac{\phi}{\pi}\right)^2 \frac{n(n-1)}{2} \end{aligned} \quad (\text{A.28})$$

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$$\begin{aligned}
\sum_{j=1}^{k-1} \sum_{i=1}^{j-1} E[\hat{I}_i^*] E[\hat{I}_j^*] \\
&= \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} E[\hat{I}_i^*] E[\hat{I}_j^*] = \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} \frac{2\phi(k-i)n}{\pi(k-1)k} \frac{2\phi(k-j)n}{\pi(k-1)k} \\
&= \left(\frac{2n\phi}{\pi}\right)^2 \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} \frac{(k-i)(k-j)}{(k-1)^2 k^2} \\
&= \left(\frac{n\phi}{\pi}\right)^2 \frac{(3k^2 - 7k + 2)}{6k(k-1)} \\
&= \left(\frac{n\phi}{\pi}\right)^2 \frac{(3k-1)(k-2)}{6k(k-1)}
\end{aligned} \tag{A.29}$$

Variance

The variance is A.1:

$$\begin{aligned}
Var[\hat{I}] &= \sum_{i=1}^{k-1} Var[\hat{I}_i^*] + 2 \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} E[\hat{I}_i^* \hat{I}_j^*] - 2 \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} E[\hat{I}_i^*] E[\hat{I}_j^*] \\
&= \left(\frac{\phi}{\pi}\right)^2 n \left(\frac{2n(3k-2)}{3k(k-1)(k-2)} - \frac{4}{3(k-2)} \right) + 2 \left(\frac{\phi}{\pi}\right)^2 \frac{n(n-1)}{2} - 2 \left(\frac{n\phi}{\pi}\right)^2 \frac{(3k-1)(k-2)}{6k(k-1)} \\
&= \left(\frac{\phi}{\pi}\right)^2 \left(n \left[\frac{2n(3k-2)}{3k(k-1)(k-2)} - \frac{4}{3(k-2)} \right] + 2 \frac{n(n-1)}{2} - 2 \frac{n^2(3k-1)(k-2)}{6k(k-1)} \right) \\
&= \left(\frac{\phi}{\pi}\right)^2 \left(n \left[\frac{2n(3k-2)}{3k(k-1)(k-2)} - \frac{4}{3(k-2)} \right] + n(n-1) - \frac{n^2(3k-1)(k-2)}{3k(k-1)} \right) \\
&= \left(\frac{\phi}{\pi}\right)^2 n \left(\frac{2n(3k-2) - 4k(k-1) + (n-1)3k(k-1)(k-2) - n(3k-1)(k-2)^2}{3k(k-1)(k-2)} \right) \\
&= \left(\frac{\phi}{\pi}\right)^2 n \left(\frac{-4nk + 4k^2n - 3k^3 + 5k^2 - 2k}{3k(k-1)(k-2)} \right) \\
&= \left(\frac{\phi}{\pi}\right)^2 n \left(\frac{4nk(k-1) - k(k-1)(3k-2)}{3k(k-1)(k-2)} \right) \\
&= \left(\frac{\phi}{\pi}\right)^2 n \left(\frac{4n - 3k + 2}{3(k-2)} \right) \\
&= I^2(P) \left(\frac{4n - 3k + 2}{3n(k-2)} \right)
\end{aligned} \tag{A.30}$$

A.5. Silverman/Quartic kernel

The variance is A.1:

$$Var[\hat{I}] = \sum_{i=1}^{k-1} Var[\hat{I}_i^*] + 2 \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} E[\hat{I}_i^* \hat{I}_j^*] - 2 \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} E[\hat{I}_i^*] E[\hat{I}_j^*] \tag{A.31}$$

$$\sum_{i=1}^{k-1} \text{Var}[\hat{I}_i^*]$$

$$\text{Var}[\hat{I}_i^*] = E[\hat{I}_i^{*2}] - (E[\hat{I}_i^*])^2 \quad (\text{A.32})$$

So we have

$$\sum_{i=1}^{k-1} \text{Var}[\hat{I}_i^*] = \sum_{i=1}^{k-1} \left(E[\hat{I}_i^{*2}] - (E[\hat{I}_i^*])^2 \right) \quad (\text{A.33})$$

A.5.0.1. $E[\hat{I}_i^{*2}]$

Since $\forall r_i > r_k, f_{X_{(i,k)}}(r_i, r_k) = 0$, the inner integral upper limit is r_k

$$\begin{aligned} E[\hat{I}_i^{*2}] &= \int_0^1 \int_0^{r_k} \hat{I}_i^{*2}(r_i, r_k) f_{X_{(i,k)}}(r_i, r_k) dr_i dr_k \\ &= \int_0^1 \int_0^{r_k} \left(\frac{3}{\pi r_k^2} \left[1 - \left(\frac{r_i}{r_k} \right)^2 \right]^2 \phi \right)^2 \mathcal{C}_K 2r_i^{2i-1} [r_k^2 - r_i^2]^{k-i-1} dr_i dr_k \\ &= \int_0^1 \int_0^{r_k} \left(\frac{3}{\pi r_k^2} \left[1 - \left(\frac{r_i}{r_k} \right)^2 \right]^2 \phi \right)^2 2C_J r_k [1 - r_k^2]^{n-k} 2r_i^{2i-1} [r_k^2 - r_i^2]^{k-i-1} dr_i dr_k \\ &= \left(\frac{3\phi}{\pi} \right)^2 C_J \int_0^1 \int_0^{r_k} \frac{1}{r_k^4} \left[1 - \left(\frac{r_i}{r_k} \right)^2 \right]^4 2r_k [1 - r_k^2]^{n-k} 2r_i^{2i-1} [r_k^2 - r_i^2]^{k-i-1} dr_i dr_k \\ &= \left(\frac{3\phi}{\pi} \right)^2 C_J \int_0^1 \frac{2r_k [1 - r_k^2]^{n-k}}{r_k^4} \int_0^{r_k} \left[1 - \left(\frac{r_i}{r_k} \right)^2 \right]^4 2r_i^{2i-1} [r_k^2 - r_i^2]^{k-i-1} dr_i dr_k \end{aligned} \quad (\text{A.34})$$

The value of the inner integral is:

$$\int_0^{r_k} \left[1 - \left(\frac{r_i}{r_k} \right)^2 \right]^4 2r_i^{2i-1} [r_k^2 - r_i^2]^{k-i-1} dr_i = \frac{r_k^{2(k-1)} \Gamma(i) \Gamma(k-i+4)}{\Gamma(k+4)} \quad (\text{A.35})$$

Then we can solve the outer integral:

$$\int_0^1 \frac{2r_k [1 - r_k^2]^{n-k}}{r_k^4} \frac{r_k^{2(k-1)} \Gamma(i) \Gamma(k-i+4)}{\Gamma(k+4)} dr_k = \frac{\Gamma(i) \Gamma(k-2) \Gamma(k-i+4) \Gamma(n-k+1)}{\Gamma(k+4) \Gamma(n-1)} \quad (\text{A.36})$$

Finally, we have:

$$\begin{aligned} E[\hat{I}_i^{*2}] &= \left(\frac{3\phi}{\pi} \right)^2 C_J \frac{\Gamma(i) \Gamma(k-2) \Gamma(k-i+4) \Gamma(n-k+1)}{\Gamma(k+4) \Gamma(n-1)} \\ &= \left(\frac{3\phi}{\pi} \right)^2 \frac{n!}{(i-1)!(k-i-1)!(n-k)!} \frac{\Gamma(i) \Gamma(k-2) \Gamma(k-i+4) \Gamma(n-k+1)}{\Gamma(k+4) \Gamma(n-1)} \\ &= \left(\frac{3\phi}{\pi} \right)^2 \frac{n!}{(i-1)!(k-i-1)!(n-k)!} \frac{(i-1)!(k-3)!(k-i+3)!(n-k)!}{(k+3)!(n-2)!} \\ &= \left(\frac{3\phi}{\pi} \right)^2 \frac{n(n-1)(k-i+3)(k-i+2)(k-i+1)(k-i)}{(k+3)(k+2)(k+1)k(k-1)(k-2)} \end{aligned} \quad (\text{A.37})$$

A.5.0.2. $\sum_{i=1}^{k-1} \left(E[\hat{I}_i^{*2}] - (E[\hat{I}_i^*])^2 \right)$

From García et al. [GHUPS14], we already know that:

$$E[\hat{I}_i^*] = \frac{3\phi}{\pi} \frac{(k-i+1)(k-i)n}{(k+1)(k-1)k} \quad (\text{A.38})$$

So we have now

$$\begin{aligned} \sum_{i=1}^{k-1} \text{Var}[\hat{I}_i^*] &= \sum_{i=1}^{k-1} \left(E[\hat{I}_i^{*2}] - (E[\hat{I}_i^*])^2 \right) \\ &= \sum_{i=1}^{k-1} \left(\left(\frac{3\phi}{\pi} \right)^2 \frac{n(n-1)(k-i+3)(k-i+2)(k-i+1)(k-i)}{(k+3)(k+2)(k+1)k(k-1)(k-2)} \right. \\ &\quad \left. - \left(\frac{3\phi}{\pi} \frac{(k-i+1)(k-i)n}{(k+1)(k-1)k} \right)^2 \right) \\ &= \left(\frac{3\phi}{\pi} \right)^2 \sum_{i=1}^{k-1} \left(\frac{n(n-1)(k-i+3)(k-i+2)(k-i+1)(k-i)}{(k+3)(k+2)(k+1)k(k-1)(k-2)} \right. \\ &\quad \left. - \left(\frac{(k-i+1)(k-i)n}{(k+1)(k-1)k} \right)^2 \right) \quad (\text{A.39}) \\ &= \left(\frac{\phi}{\pi} \right)^2 \frac{3n(-k[3k(k-2n) + n-3] - 4n)}{5(k+1)k(k-1)(k-2)} \\ &= I^2(P) \frac{3(-k[3k(k-2n) + n-3] - 4n)}{5n(k+1)k(k-1)(k-2)} \\ &= I^2(P) \frac{-3(3k^3 - 3k) - 3n(-6k^2 + k + 4)}{5n(k+1)k(k-1)(k-2)} \\ &= I^2(P) \left(\frac{3(6k^2 - k - 4)}{5(k+1)k(k-1)(k-2)} - \frac{9}{5n(k-2)} \right) \end{aligned}$$

$$\sum_{j=1}^{k-1} \sum_{i=1}^{j-1} E[\hat{I}_i^* \hat{I}_j^*]$$

$$\begin{aligned} E[\hat{I}_i^* \hat{I}_j^*] &= \int_0^1 \int_0^{r_k} \int_0^{r_j} \hat{I}_i^*(r_i, r_k) \hat{I}_j^*(r_j, r_k) f_{X(i,j,k)}(r_i, r_j, r_k) dr_i dr_j dr_k \\ &= \int_0^1 \int_0^{r_k} \int_0^{r_j} \left(\frac{3}{\pi r_k^2} \left[1 - \left(\frac{r_i}{r_k} \right)^2 \right]^2 \phi \right) \left(\frac{3}{\pi r_k^2} \left[1 - \left(\frac{r_j}{r_k} \right)^2 \right]^2 \phi \right) \times \\ &\quad \mathcal{C}_{\textcircled{2}} 2r_i 2r_j 2r_k [r_i^2]^{i-1} [r_j^2 - r_i^2]^{j-i-1} [r_k^2 - r_j^2]^{k-j-1} [1 - r_k^2]^{n-k} dr_i dr_j dr_k \\ &= \left(\frac{3\phi}{\pi} \right)^2 \mathcal{C}_{\textcircled{2}} 2^3 \int_0^1 \int_0^{r_k} \int_0^{r_j} \frac{1}{r_k^4} \left[1 - \left(\frac{r_i}{r_k} \right)^2 \right]^2 \left[1 - \left(\frac{r_j}{r_k} \right)^2 \right]^2 \times \\ &\quad r_i r_j r_k [r_i^2]^{i-1} [r_j^2 - r_i^2]^{j-i-1} [r_k^2 - r_j^2]^{k-j-1} [1 - r_k^2]^{n-k} dr_i dr_j dr_k \quad (\text{A.40}) \\ &= 8 \left(\frac{3\phi}{\pi} \right)^2 \mathcal{C}_{\textcircled{2}} \int_0^1 \frac{[1 - r_k^2]^{n-k}}{r_k^3} \int_0^{r_k} \left[1 - \left(\frac{r_j}{r_k} \right)^2 \right]^2 r_j [r_k^2 - r_j^2]^{k-j-1} \\ &\quad \int_0^{r_j} \left[1 - \left(\frac{r_i}{r_k} \right)^2 \right]^2 r_i [r_i^2]^{i-1} [r_j^2 - r_i^2]^{j-i-1} dr_i dr_j dr_k \end{aligned}$$

The value of the inner integral is:

$$\begin{aligned}
& \int_0^{r_j} \left[1 - \left(\frac{r_i}{r_k}\right)^2\right]^2 r_i [r_i^2]^{i-1} [r_j^2 - r_i^2]^{j-i-1} dr_i \\
&= \int_0^{r_j} \left[1 - \left(\frac{r_i}{r_k}\right)^2\right]^2 [r_i]^{2i-1} [r_j^2 - r_i^2]^{j-i-1} dr_i \\
&= \frac{r_j^{2(j-1)}(r_k^4 j(j+1) + r_j^4 i(i+1) - 2r_k^2 r_j^2 i(j+1))\Gamma[i]\Gamma[j-i]}{2r_k^4 \Gamma[j+2]}
\end{aligned} \tag{A.41}$$

Then, the value of the middle integral is:

$$\begin{aligned}
& \int_0^{r_k} \left[1 - \left(\frac{r_j}{r_k}\right)^2\right]^2 r_j [r_k^2 - r_j^2]^{k-j-1} \times \\
& \frac{r_j^{2(j-1)}(r_k^4 j(j+1) + r_j^4 i(i+1) - 2r_k^2 r_j^2 i(j+1))\Gamma[i]\Gamma[j-i]}{2r_k^4 \Gamma[j+2]} dr_j \\
&= \frac{(k-i+3)(k-i+2)r_k^{2(k-1)}\Gamma[i]\Gamma[j-i]\Gamma[k-j+2]}{4\Gamma[k+4]}
\end{aligned} \tag{A.42}$$

Then we can solve the outer integral:

$$\begin{aligned}
& \int_0^1 \frac{[1 - r_k^2]^{n-k}}{r_k^3} \frac{(k-i+3)(k-i+2)r_k^{2(k-1)}\Gamma[i]\Gamma[j-i]\Gamma[k-j+2]}{4\Gamma[k+4]} dr_k \\
&= \frac{(k-i+3)(k-i+2)\Gamma[i]\Gamma[j-i]\Gamma[k-2]\Gamma[k-j+2]\Gamma[n-k+1]}{8\Gamma[k+4]\Gamma[n-1]}
\end{aligned} \tag{A.43}$$

Then, we have:

$$\begin{aligned}
E[\hat{I}_i^* \hat{I}_j^*] &= 8 \left(\frac{3\phi}{\pi}\right)^2 \mathcal{C}_{\textcircled{3}} \frac{(k-i+3)(k-i+2)\Gamma[i]\Gamma[j-i]\Gamma[k-2]\Gamma[k-j+2]\Gamma[n-k+1]}{8\Gamma[k+4]\Gamma[n-1]} \\
&= 8 \left(\frac{3\phi}{\pi}\right)^2 \frac{n!}{(i-1)!(j-i-1)!(k-j-1)!(n-k)!} \times \\
& \quad \frac{(k-i+3)(k-i+2)\Gamma[i]\Gamma[j-i]\Gamma[k-2]\Gamma[k-j+2]\Gamma[n-k+1]}{8\Gamma[k+4]\Gamma[n-1]} \\
&= 8 \left(\frac{3\phi}{\pi}\right)^2 \frac{n!}{(i-1)!(j-i-1)!(k-j-1)!(n-k)!} \times \\
& \quad \frac{(k-i+3)(k-i+2)(i-1)!(j-i-1)!(k-3)!(k-j+1)!(n-k)!}{8(k+3)!(n-2)!} \\
&= \left(\frac{3\phi}{\pi}\right)^2 \frac{n(n-1)(k-i+3)(k-i+2)(k-j+1)(k-j)}{(k+3)(k+2)(k+1)k(k-1)(k-2)}
\end{aligned} \tag{A.44}$$

A. Appendix

Finally, we have:

$$\begin{aligned}
\sum_{j=1}^{k-1} \sum_{i=1}^{j-1} E[\hat{I}_i^* \hat{I}_j^*] &= \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} \left(\left(\frac{3\phi}{\pi} \right)^2 \frac{n(n-1)(k-i+3)(k-i+2)(k-j+1)(k-j)}{(k+3)(k+2)(k+1)k(k-1)(k-2)} \right) \\
&= 9n(n-1) \left(\frac{\phi}{\pi} \right)^2 \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} \left(\frac{(k-i+3)(k-i+2)(k-j+1)(k-j)}{(k+3)(k+2)(k+1)k(k-1)(k-2)} \right) \\
&= \left(\frac{\phi}{\pi} \right)^2 \frac{n(n-1)}{2} \\
&= I^2(P) \frac{(n-1)}{2n}
\end{aligned} \tag{A.45}$$

$$\begin{aligned}
\sum_{j=1}^{k-1} \sum_{i=1}^{j-1} E[\hat{I}_i^*] E[\hat{I}_j^*] &= \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} \frac{3\phi}{\pi} \frac{(k-i+1)(k-i)n}{(k+1)(k-1)k} \frac{3\phi}{\pi} \frac{(k-j+1)(k-j)n}{(k+1)(k-1)k} \\
&= \left(\frac{3n\phi}{\pi} \right)^2 \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} \frac{(k-i+1)(k-i)(k-j+1)(k-j)}{(k^2-1)^2 k^2} \\
&= \left(\frac{n\phi}{\pi} \right)^2 \frac{(k-2)(5k^2+k-3)}{10k(k^2-1)} \\
&= I^2(P) \left(\frac{(k-2)(5k^2+k-3)}{10k(k-1)(k+1)} \right)
\end{aligned} \tag{A.46}$$

Variance

The variance is A.1:

$$\begin{aligned}
Var[\hat{I}] &= \sum_{i=1}^{k-1} Var[\hat{I}_i^*] + 2 \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} E[\hat{I}_i^* \hat{I}_j^*] - 2 \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} E[\hat{I}_i^*] E[\hat{I}_j^*] \\
&= I^2(P) \left(\frac{3(6k^2-k-4)}{5(k+1)k(k-1)(k-2)} - \frac{9}{5n(k-2)} \right) + 2I^2(P) \frac{(n-1)}{2n} - 2I^2(P) \left(\frac{(k-2)(5k^2+k-3)}{10k(k-1)(k+1)} \right) \\
&= I^2(P) \left(\frac{3(6k^2-k-4)}{5(k+1)k(k-1)(k-2)} - \frac{9}{5n(k-2)} + \frac{(n-1)}{n} - \frac{(k-2)(5k^2+k-3)}{5k(k-1)(k+1)} \right) \\
&= I^2(P) \frac{3n(6k^2-k-4) - 9k(k^2-1) + 5(k+1)k(k-1)(k-2)(n-1) - n(k-2)^2(5k^2+k-3)}{5n(k+1)k(k-1)(k-2)} \\
&= I^2(P) \frac{-9kn - 5k^4 + k^3 + 5k^2 - k + 9k^3n}{5n(k+1)k(k-1)(k-2)} \\
&= I^2(P) \frac{9nk(k^2-1) + k(k^2-1) - 5k^2(k^2-1)}{5n(k+1)k(k-1)(k-2)} \\
&= I^2(P) \frac{9n - 5k + 1}{5n(k-2)}
\end{aligned} \tag{A.47}$$

A.6. Corrected photon mapping with cone filter

The variance is A.1:

$$Var[\hat{I}] = \sum_{i=1}^{k-1} Var[\hat{I}_i^*] + 2 \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} E[\hat{I}_i^* \hat{I}_j^*] - 2 \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} E[\hat{I}_i^*] E[\hat{I}_j^*] \quad (\text{A.48})$$

$$\sum_{i=1}^{k-1} Var[\hat{I}_i^*]$$

$$Var[\hat{I}_i^*] = E[\hat{I}_i^{*2}] - (E[\hat{I}_i^*])^2 \quad (\text{A.49})$$

So we have

$$\sum_{i=1}^{k-1} Var[\hat{I}_i^*] = \sum_{i=1}^{k-1} \left(E[\hat{I}_i^{*2}] - (E[\hat{I}_i^*])^2 \right) \quad (\text{A.50})$$

A.6.0.1. $E[\hat{I}_i^{*2}]$

Since $\forall r_i > r_k, f_{X_{(i,k)}}(r_i, r_k) = 0$, the inner integral upper limit is r_k

$$\begin{aligned} E[\hat{I}_i^{*2}] &= \int_0^1 \int_0^{r_k} \hat{I}_i^{*2}(r_i, r_k) f_{X_{(i,k)}}(r_i, r_k) dr_i dr_k \\ &= \int_0^1 \int_0^{r_k} \left(\frac{1 - \frac{r_i}{sr_k}}{\pi r_k^2 \left(1 - \frac{2}{3s}\right)} \phi \right)^2 C_K 2r_i^{2i-1} [r_k^2 - r_i^2]^{k-i-1} dr_i dr_k \\ &= \int_0^1 \int_0^{r_k} \left(\frac{1 - \frac{r_i}{sr_k}}{\pi r_k^2 \left(1 - \frac{2}{3s}\right)} \phi \right)^2 2C_J r_k [1 - r_k^2]^{n-k} 2r_i^{2i-1} [r_k^2 - r_i^2]^{k-i-1} dr_i dr_k \\ &= \left(\frac{\phi}{\pi \left(1 - \frac{2}{3s}\right)} \right)^2 C_J \int_0^1 \int_0^{r_k} \frac{1}{r_k^4} \left(1 - \frac{r_i}{sr_k}\right)^2 2r_k [1 - r_k^2]^{n-k} 2r_i^{2i-1} [r_k^2 - r_i^2]^{k-i-1} dr_i dr_k \\ &= 4 \left(\frac{3s\phi}{\pi(3s-2)} \right)^2 C_J \int_0^1 \frac{r_k [1 - r_k^2]^{n-k}}{r_k^4} \int_0^{r_k} \left(1 - \frac{r_i}{sr_k}\right)^2 r_i^{2i-1} [r_k^2 - r_i^2]^{k-i-1} dr_i dr_k \end{aligned} \quad (\text{A.51})$$

The value of the inner integral is:

$$\int_0^{r_k} \left(1 - \frac{r_i}{sr_k}\right)^2 r_i^{2i-1} [r_k^2 - r_i^2]^{k-i-1} dr_i = \frac{r_k^{2(k-1)} \Gamma(k-i)}{2s^2} \left(\frac{(ks^2 + i)\Gamma(i)}{\Gamma(k+1)} - \frac{2s\Gamma(i + \frac{1}{2})}{\Gamma(k + \frac{1}{2})} \right) \quad (\text{A.52})$$

Then we can solve the outer integral:

$$\begin{aligned} &\int_0^1 \frac{r_k [1 - r_k^2]^{n-k}}{r_k^4} \frac{r_k^{2(k-1)} \Gamma(k-i)}{2s^2} \left(\frac{(ks^2 + i)\Gamma(i)}{\Gamma(k+1)} - \frac{2s\Gamma(i + \frac{1}{2})}{\Gamma(k + \frac{1}{2})} \right) dr_k \\ &= \frac{\Gamma(k-2)\Gamma(k-i)\Gamma(n-k+1)}{4s^2\Gamma(n-1)} \left(\frac{(ks^2 + i)\Gamma(i)}{\Gamma(k+1)} - \frac{2s\Gamma(i + \frac{1}{2})}{\Gamma(k + \frac{1}{2})} \right) \end{aligned} \quad (\text{A.53})$$

A. Appendix

Finally, we have:

$$\begin{aligned}
E[\hat{I}_i^{*2}] &= 4 \left(\frac{3s\phi}{\pi(3s-2)} \right)^2 C_J \frac{\Gamma(k-2)\Gamma(k-i)\Gamma(n-k+1)}{4s^2\Gamma(n-1)} \left(\frac{(ks^2+i)\Gamma(i)}{\Gamma(k+1)} - \frac{2s\Gamma(i+\frac{1}{2})}{\Gamma(k+\frac{1}{2})} \right) \\
&= 4 \left(\frac{3s\phi}{\pi(3s-2)} \right)^2 \frac{n!}{(i-1)!(k-i-1)!(n-k)!} \times \\
&\quad \frac{\Gamma(k-2)\Gamma(k-i)\Gamma(n-k+1)}{4s^2\Gamma(n-1)} \left(\frac{(ks^2+i)\Gamma(i)}{\Gamma(k+1)} - \frac{2s\Gamma(i+\frac{1}{2})}{\Gamma(k+\frac{1}{2})} \right) \\
&= 4 \left(\frac{3s\phi}{\pi(3s-2)} \right)^2 \frac{n!}{(i-1)!(k-i-1)!(n-k)!} \times \\
&\quad \frac{(k-3)!(k-i-1)!(n-k)!}{4s^2(n-2)!} \left(\frac{(ks^2+i)(i-1)!}{k!} - \frac{2s\Gamma(i+\frac{1}{2})}{\Gamma(k+\frac{1}{2})} \right) \\
&= 4 \left(\frac{3s\phi}{\pi(3s-2)} \right)^2 \frac{n(n-1)(k-3)!}{4s^2(i-1)!} \left(\frac{(ks^2+i)(i-1)!}{k!} - \frac{2s\Gamma(i+\frac{1}{2})}{\Gamma(k+\frac{1}{2})} \right) \\
&= 4 \left(\frac{3s\phi}{\pi(3s-2)} \right)^2 \frac{n(n-1)}{4s^2} \left(\frac{(k-3)!(ks^2+i)(i-1)!}{(i-1)!k!} - \frac{(k-3)!2s\Gamma(i+\frac{1}{2})}{(i-1)!\Gamma(k+\frac{1}{2})} \right) \\
&= \left(\frac{3\phi}{\pi(3s-2)} \right)^2 n(n-1) \left(\frac{(ks^2+i)}{k(k-1)(k-2)} - \frac{(k-3)!2s\Gamma(i+\frac{1}{2})}{(i-1)!\Gamma(k+\frac{1}{2})} \right) \\
&= \left(\frac{3\phi}{\pi(3s-2)} \right)^2 n(n-1) \left(\frac{ks^2+i}{k(k-1)(k-2)} - \frac{2s\Gamma(k-2)\Gamma(i+\frac{1}{2})}{\Gamma(i)\Gamma(k+\frac{1}{2})} \right) \\
&= I^2(P) \frac{9(n-1)}{n(3s-2)^2} \left(\frac{ks^2+i}{k(k-1)(k-2)} - \frac{2s\Gamma(k-2)\Gamma(i+\frac{1}{2})}{\Gamma(i)\Gamma(k+\frac{1}{2})} \right)
\end{aligned} \tag{A.54}$$

A.6.0.2. $\sum_{i=1}^{k-1} (E[\hat{I}_i^{*2}] - (E[\hat{I}_i^*])^2)$

From García et al. [GHUPS14], we already know that:

$$\begin{aligned}
E[\hat{I}_i^*] &= \frac{\phi}{\pi(1-\frac{2}{3s})} n \left(\frac{1}{k-1} - \frac{\Gamma(i+\frac{1}{2})\Gamma(k-1)}{s\Gamma(i)\Gamma(k+\frac{1}{2})} \right) \\
&= \frac{3s\phi}{(3s-2)\pi} n \left(\frac{1}{k-1} - \frac{\Gamma(i+\frac{1}{2})\Gamma(k-1)}{s\Gamma(i)\Gamma(k+\frac{1}{2})} \right) \\
&= I(P) \left(\frac{3s}{3s-2} \right) \left(\frac{1}{k-1} - \frac{\Gamma(i+\frac{1}{2})\Gamma(k-1)}{s\Gamma(i)\Gamma(k+\frac{1}{2})} \right)
\end{aligned} \tag{A.55}$$

So we have now

$$\begin{aligned}
& \sum_{i=1}^{k-1} \text{Var}[\hat{I}_i^*] = \sum_{i=1}^{k-1} \left(E[\hat{I}_i^{*2}] - (E[\hat{I}_i^*])^2 \right) \\
&= \sum_{i=1}^{k-1} I^2(P) \frac{9(n-1)}{n(3s-2)^2} \left(\frac{ks^2+i}{k(k-1)(k-2)} - \frac{2s\Gamma(k-2)\Gamma(i+\frac{1}{2})}{\Gamma(i)\Gamma(k+\frac{1}{2})} \right) \\
&\quad - \sum_{i=1}^{k-1} \left(I(P) \left(\frac{3s}{3s-2} \right) \left(\frac{1}{k-1} - \frac{\Gamma(i+\frac{1}{2})\Gamma(k-1)}{s\Gamma(i)\Gamma(k+\frac{1}{2})} \right) \right)^2 \\
&= \sum_{i=1}^{k-1} I^2(P) \frac{9(n-1)}{n(3s-2)^2} \left(\frac{ks^2+i}{k(k-1)(k-2)} - \frac{2s\Gamma(k-2)\Gamma(i+\frac{1}{2})}{\Gamma(i)\Gamma(k+\frac{1}{2})} \right) \\
&\quad - \sum_{i=1}^{k-1} I^2(P) \left(\frac{9s^2}{(3s-2)^2} \right) \left(\frac{1}{k-1} - \frac{\Gamma(i+\frac{1}{2})\Gamma(k-1)}{s\Gamma(i)\Gamma(k+\frac{1}{2})} \right)^2 \\
&= I^2(P) \frac{3}{2(3s-2)^2} \left(\frac{6(n-k+1)s^2 - 8(n-k+1)s + 3(n-1)(k-1)}{n(k-1)(k-2)} - \frac{6[\Gamma(k-1)]^2}{[\Gamma(k+\frac{1}{2})]^2} u(k) \right) \\
&= I^2(P) \left[\frac{n-k+1}{n(k-1)(k-2)} + \frac{1}{(3s-2)^2} \left(\frac{9k-17}{2(k-1)(k-2)} - \frac{9[\Gamma(k-1)]^2}{[\Gamma(k+\frac{1}{2})]^2} u(k) - \frac{1}{2n(k-2)} \right) \right]
\end{aligned} \tag{A.56}$$

where function $u(k)$ with $k \in \mathbb{N}$ defined as:

$$\begin{aligned}
u(1) &= 0 & u(2) &= \frac{\pi}{4} \\
u(k) &= \frac{(8k^2 - 28k + 25)u(k-1) - (2k-3)^2 u(k-2)}{4(k-2)^2}
\end{aligned} \tag{A.57}$$

$$\sum_{j=1}^{k-1} \sum_{i=1}^{j-1} E[\hat{I}_i^* \hat{I}_j^*]$$

$$\begin{aligned}
E[\hat{I}_i^* \hat{I}_j^*] &= \int_0^1 \int_0^{r_k} \int_0^{r_j} \hat{I}_i^*(r_i, r_k) \hat{I}_j^*(r_j, r_k) f_{X_{(i,j,k)}}(r_i, r_j, r_k) dr_i dr_j dr_k \\
&= \int_0^1 \int_0^{r_k} \int_0^{r_j} \left(\frac{1 - \frac{r_i}{sr_k}}{\pi r_k^2 (1 - \frac{2}{3s})} \phi \right) \left(\frac{1 - \frac{r_j}{sr_k}}{\pi r_k^2 (1 - \frac{2}{3s})} \phi \right) \times \\
&\quad \mathcal{C}_{\textcircled{2}} 2r_i 2r_j 2r_k [r_i^2]^{i-1} [r_j^2 - r_i^2]^{j-i-1} [r_k^2 - r_j^2]^{k-j-1} [1 - r_k^2]^{n-k} dr_i dr_j dr_k \\
&= \left(\frac{\phi}{\pi(1 - \frac{2}{3s})} \right)^2 \mathcal{C}_{\textcircled{2}}^3 \int_0^1 \int_0^{r_k} \int_0^{r_j} \frac{1}{r_k^4} \left(1 - \frac{r_i}{sr_k} \right) \left(1 - \frac{r_j}{sr_k} \right) \times \\
&\quad r_i r_j r_k [r_i^2]^{i-1} [r_j^2 - r_i^2]^{j-i-1} [r_k^2 - r_j^2]^{k-j-1} [1 - r_k^2]^{n-k} dr_i dr_j dr_k \\
&= 8 \left(\frac{3s\phi}{\pi(3s-2)} \right)^2 \mathcal{C}_{\textcircled{2}} \int_0^1 \frac{[1 - r_k^2]^{n-k}}{r_k^3} \int_0^{r_k} \left(1 - \frac{r_j}{sr_k} \right) r_j [r_k^2 - r_j^2]^{k-j-1} \\
&\quad \int_0^{r_j} \left(1 - \frac{r_i}{sr_k} \right) r_i [r_i^2]^{i-1} [r_j^2 - r_i^2]^{j-i-1} dr_i dr_j dr_k
\end{aligned} \tag{A.58}$$

A. Appendix

The value of the inner integral is:

$$\begin{aligned}
& \int_0^{r_j} \left(1 - \frac{r_i}{sr_k}\right) r_i [r_i^2]^{i-1} [r_j^2 - r_i^2]^{j-i-1} dr_i \\
&= \int_0^{r_j} \left(1 - \frac{r_i}{sr_k}\right) [r_i]^{2i-1} [r_j^2 - r_i^2]^{j-i-1} dr_i \\
&= \frac{r_j^{2(j-1)} [sr_k \Gamma(i) \Gamma(j + \frac{1}{2}) - r_j \Gamma(j) \Gamma(i + \frac{1}{2})] \Gamma(j-i)}{2sr_k \Gamma(j) \Gamma(j + \frac{1}{2})}
\end{aligned} \tag{A.59}$$

Then, the value of the middle integral is:

$$\begin{aligned}
& \int_0^{r_k} \left(1 - \frac{r_j}{sr_k}\right) r_j [r_k^2 - r_j^2]^{k-j-1} \frac{r_j^{2(j-1)} [sr_k \Gamma(i) \Gamma(j + \frac{1}{2}) - r_j \Gamma(j) \Gamma(i + \frac{1}{2})] \Gamma[j-i]}{2sr_k \Gamma(j) \Gamma(j + \frac{1}{2})} dr_j \\
&= \frac{r_k^{2(k-1)} \Gamma(j-i) \Gamma(k-j)}{4s^2 \Gamma(j) \Gamma(j + \frac{1}{2}) \Gamma(k) \Gamma(k + \frac{1}{2}) \Gamma(k+1)} (s\Gamma(i) \Gamma(j + \frac{1}{2}) \Gamma(k+1) [s\Gamma(j) \Gamma(k + \frac{1}{2}) - \Gamma(k) \Gamma(j + \frac{1}{2})] \\
& \quad + \Gamma(i + \frac{1}{2}) \Gamma(j) \Gamma(k) [\Gamma(j+1) \Gamma(k + \frac{1}{2}) - s\Gamma(j + \frac{1}{2}) \Gamma(k+1)])
\end{aligned} \tag{A.60}$$

Then we can solve the outer integral:

$$\begin{aligned}
& \int_0^1 \frac{[1 - r_k^2]^{n-k}}{r_k^3} \frac{r_k^{2(k-1)} \Gamma(j-i) \Gamma(k-j)}{4s^2 \Gamma(j) \Gamma(j + \frac{1}{2}) \Gamma(k) \Gamma(k + \frac{1}{2}) \Gamma(k+1)} \times \\
& \quad (s\Gamma(i) \Gamma(j + \frac{1}{2}) \Gamma(k+1) [s\Gamma(j) \Gamma(k + \frac{1}{2}) - \Gamma(k) \Gamma(j + \frac{1}{2})] \\
& \quad + \Gamma(i + \frac{1}{2}) \Gamma(j) \Gamma(k) [\Gamma(j+1) \Gamma(k + \frac{1}{2}) - s\Gamma(j + \frac{1}{2}) \Gamma(k+1)]) dr_k \\
&= \frac{\Gamma(j-i) \Gamma(k-2) \Gamma(k-j) \Gamma(n-k+1)}{8s^2 \Gamma(j) \Gamma(j + \frac{1}{2}) \Gamma(k) \Gamma(k + \frac{1}{2}) \Gamma(k+1) \Gamma(n-1)} \times \\
& \quad (s\Gamma(i) \Gamma(j + \frac{1}{2}) \Gamma(k+1) [s\Gamma(j) \Gamma(k + \frac{1}{2}) - \Gamma(k) \Gamma(j + \frac{1}{2})] \\
& \quad + \Gamma(i + \frac{1}{2}) \Gamma(j) \Gamma(k) [\Gamma(j+1) \Gamma(k + \frac{1}{2}) - s\Gamma(j + \frac{1}{2}) \Gamma(k+1)])
\end{aligned} \tag{A.61}$$

Then, we have:

$$\begin{aligned}
E[\hat{I}_i^* \hat{I}_j^*] &= 8 \left(\frac{3s\phi}{\pi(3s-2)} \right)^2 \mathcal{C}_{\oplus} \frac{\Gamma(j-i)\Gamma(k-2)\Gamma(k-j)\Gamma(n-k+1)}{8s^2\Gamma(j)\Gamma(j+\frac{1}{2})\Gamma(k)\Gamma(k+\frac{1}{2})\Gamma(k+1)\Gamma(n-1)} \times \\
&\quad (s\Gamma(i)\Gamma(j+\frac{1}{2})\Gamma(k+1)[s\Gamma(j)\Gamma(k+\frac{1}{2}) - \Gamma(k)\Gamma(j+\frac{1}{2})] \\
&\quad + \Gamma(i+\frac{1}{2})\Gamma(j)\Gamma(k)[\Gamma(j+1)\Gamma(k+\frac{1}{2}) - s\Gamma(j+\frac{1}{2})\Gamma(k+1)]) \\
&= 8 \left(\frac{3s\phi}{\pi(3s-2)} \right)^2 \frac{n!}{(i-1)!(j-i-1)!(k-j-1)!(n-k)!} \times \\
&\quad \frac{\Gamma(j-i)\Gamma(k-2)\Gamma(k-j)\Gamma(n-k+1)}{8s^2\Gamma(j)\Gamma(j+\frac{1}{2})\Gamma(k)\Gamma(k+\frac{1}{2})\Gamma(k+1)\Gamma(n-1)} \times \\
&\quad (s\Gamma(i)\Gamma(j+\frac{1}{2})\Gamma(k+1)[s\Gamma(j)\Gamma(k+\frac{1}{2}) - \Gamma(k)\Gamma(j+\frac{1}{2})] \\
&\quad + \Gamma(i+\frac{1}{2})\Gamma(j)\Gamma(k)[\Gamma(j+1)\Gamma(k+\frac{1}{2}) - s\Gamma(j+\frac{1}{2})\Gamma(k+1)]) \\
&= 8 \left(\frac{3s\phi}{\pi(3s-2)} \right)^2 \frac{n!}{(i-1)!(j-i-1)!(k-j-1)!(n-k)!} \times \\
&\quad \frac{(j-i-1)!(k-3)!(k-j-1)!(n-k)!}{8s^2(j-1)!\Gamma(j+\frac{1}{2})(k-1)!\Gamma(k+\frac{1}{2})k!(n-2)!} \times \\
&\quad (s\Gamma(i)\Gamma(j+\frac{1}{2})\Gamma(k+1)[s\Gamma(j)\Gamma(k+\frac{1}{2}) - \Gamma(k)\Gamma(j+\frac{1}{2})] \\
&\quad + \Gamma(i+\frac{1}{2})\Gamma(j)\Gamma(k)[\Gamma(j+1)\Gamma(k+\frac{1}{2}) - s\Gamma(j+\frac{1}{2})\Gamma(k+1)]) \\
&= \left(\frac{3\phi}{\pi(3s-2)} \right)^2 \frac{n(n-1)}{(k-1)(k-2)(i-1)!(j-1)!k!\Gamma(j+\frac{1}{2})\Gamma(k+\frac{1}{2})} \times \\
&\quad (s\Gamma(i)\Gamma(j+\frac{1}{2})\Gamma(k+1)[s\Gamma(j)\Gamma(k+\frac{1}{2}) - \Gamma(k)\Gamma(j+\frac{1}{2})] \\
&\quad + \Gamma(i+\frac{1}{2})\Gamma(j)\Gamma(k)[\Gamma(j+1)\Gamma(k+\frac{1}{2}) - s\Gamma(j+\frac{1}{2})\Gamma(k+1)]) \\
&= \left(\frac{3\phi}{\pi(3s-2)} \right)^2 \frac{n(n-1)}{(i-1)!} \left(\frac{ks^2\Gamma(i)\Gamma(j+\frac{1}{2}) + \Gamma(i+\frac{1}{2})\Gamma(j+1)}{k(k-1)(k-2)\Gamma(j+\frac{1}{2})} \right. \\
&\quad \left. - \frac{s\Gamma(k-2)[\Gamma(i+\frac{1}{2})\Gamma(j) + \Gamma(i)\Gamma(j+\frac{1}{2})]}{\Gamma(j)\Gamma(k+\frac{1}{2})} \right)
\end{aligned} \tag{A.62}$$

A. Appendix

Finally, we have:

$$\begin{aligned}
\sum_{j=1}^{k-1} \sum_{i=1}^{j-1} E[\hat{I}_i^* \hat{I}_j^*] &= \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} \frac{9\phi^2}{\pi^2(3s-2)^2} \frac{n(n-1)}{(i-1)!} \left(\frac{ks^2\Gamma(i)\Gamma(j+\frac{1}{2}) + \Gamma(i+\frac{1}{2})\Gamma(j+1)}{k(k-1)(k-2)\Gamma(j+\frac{1}{2})} \right. \\
&\quad \left. - \frac{s\Gamma(k-2)[\Gamma(i+\frac{1}{2})\Gamma(j) + \Gamma(i)\Gamma(j+\frac{1}{2})]}{\Gamma(j)\Gamma(k+\frac{1}{2})} \right) \\
&= \frac{9n(n-1)\phi^2}{\pi^2(3s-2)^2} \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} \left(\frac{ks^2\Gamma(i)\Gamma(j+\frac{1}{2}) + \Gamma(i+\frac{1}{2})\Gamma(j+1)}{k(k-1)(k-2)\Gamma(i)\Gamma(j+\frac{1}{2})} \right. \\
&\quad \left. - \frac{s\Gamma(k-2)[\Gamma(i+\frac{1}{2})\Gamma(j) + \Gamma(i)\Gamma(j+\frac{1}{2})]}{\Gamma(i)\Gamma(j)\Gamma(k+\frac{1}{2})} \right) \\
&= \left(\frac{\phi}{\pi} \right)^2 \frac{n(n-1)}{2} \\
&= I^2(P) \frac{(n-1)}{2n}
\end{aligned} \tag{A.63}$$

$$\sum_{j=1}^{k-1} \sum_{i=1}^{j-1} E[\hat{I}_i^*] E[\hat{I}_j^*]$$

$$\begin{aligned}
&\sum_{j=1}^{k-1} \sum_{i=1}^{j-1} E[\hat{I}_i^*] E[\hat{I}_j^*] \\
&= \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} I(P) \left(\frac{3s}{3s-2} \right) \left(\frac{1}{k-1} - \frac{\Gamma(i+\frac{1}{2})\Gamma(k-1)}{s\Gamma(i)\Gamma(k+\frac{1}{2})} \right) I(P) \left(\frac{3s}{3s-2} \right) \left(\frac{1}{k-1} - \frac{\Gamma(j+\frac{1}{2})\Gamma(k-1)}{s\Gamma(j)\Gamma(k+\frac{1}{2})} \right) \\
&= I^2(P) \left(\frac{3s}{3s-2} \right)^2 \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} \left(\frac{1}{k-1} - \frac{\Gamma(i+\frac{1}{2})\Gamma(k-1)}{s\Gamma(i)\Gamma(k+\frac{1}{2})} \right) \left(\frac{1}{k-1} - \frac{\Gamma(j+\frac{1}{2})\Gamma(k-1)}{s\Gamma(j)\Gamma(k+\frac{1}{2})} \right) \\
&= \frac{3}{2} I^2(P) \left(\frac{1}{3s-2} \right)^2 \left(\frac{s(3s-4)(k-2)}{k-1} + \frac{4[\Gamma(k-1)]^2}{[\Gamma(k+\frac{1}{2})]^2} (v(k) - u(k)) \right) \\
&= I^2(P) \left(\frac{k-2}{2(k-1)} - \frac{1}{(3s-2)^2} \left[\frac{2(k-2)}{k-1} - \frac{6[\Gamma(k-1)]^2}{[\Gamma(k+\frac{1}{2})]^2} (v(k) - u(k)) \right] \right)
\end{aligned} \tag{A.64}$$

where function $u(k)$ with $k \in \mathbb{N}$ defined in A.57 and $v(k)$ with $k \in \mathbb{N}$ defined as:

$$\begin{aligned}
v(1) &= 0 & v(2) &= \frac{\pi}{4} \\
v(k) &= \frac{(8k^3 - 40k^2 + 69k - 41)v(k-1) - (k-1)(2k-3)^2 v(k-2)}{4(k-2)^3}
\end{aligned} \tag{A.65}$$

Variance

The variance is A.1:

$$\begin{aligned}
Var[\hat{I}] &= \sum_{i=1}^{k-1} Var[\hat{I}_i^*] + 2 \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} E[\hat{I}_i^* \hat{I}_j^*] - 2 \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} E[\hat{I}_i^*] E[\hat{I}_j^*] \\
&= I^2(P) \left[\frac{n-k+1}{n(k-1)(k-2)} + \frac{1}{(3s-2)^2} \left(\frac{9k-17}{2(k-1)(k-2)} - \frac{9[\Gamma(k-1)]^2}{[\Gamma(k+\frac{1}{2})]^2} u(k) - \frac{1}{2n(k-2)} \right) \right] \\
&\quad + 2I^2(P) \frac{(n-1)}{2n} - 2I^2(P) \left(\frac{k-2}{2(k-1)} - \frac{1}{(3s-2)^2} \left[\frac{2(k-2)}{k-1} - \frac{6[\Gamma(k-1)]^2}{[\Gamma(k+\frac{1}{2})]^2} (v(k) - u(k)) \right] \right) \\
&= I^2(P) \left[\frac{n-k+1}{n(k-1)(k-2)} + \frac{(n-1)}{n} - \frac{k-2}{k-1} \right] \\
&\quad + I^2(P) \frac{1}{(3s-2)^2} \left[\frac{9k-17}{2(k-1)(k-2)} - \frac{1}{2n(k-2)} + \frac{4(k-2)}{k-1} \right] \\
&\quad + I^2(P) \frac{1}{(3s-2)^2} \left[-\frac{9[\Gamma(k-1)]^2}{[\Gamma(k+\frac{1}{2})]^2} u(k) - \frac{12[\Gamma(k-1)]^2}{[\Gamma(k+\frac{1}{2})]^2} (v(k) - u(k)) \right] \\
&= I^2(P) \frac{n-k+1 + (n-1)(k-1)(k-2) - n(k-2)^2}{n(k-1)(k-2)} \\
&\quad + I^2(P) \frac{1}{(3s-2)^2} \left[\frac{n(9k-17) - (k-1) + 4(k-2)^2 2n}{2n(k-1)(k-2)} + \frac{[\Gamma(k-1)]^2}{[\Gamma(k+\frac{1}{2})]^2} (3u(k) - 12v(k)) \right] \\
&= I^2(P) \frac{n-k+1 + k^2n - k^2 - 3kn + 3k + 2n - 2 - nk^2 + 4nk - 4n}{n(k-1)(k-2)} \\
&\quad + I^2(P) \frac{1}{(3s-2)^2} \left[\frac{9nk - 17n - k + 1 + 8k^2n - 32kn + 32n}{2n(k-1)(k-2)} + \frac{3[\Gamma(k-1)]^2}{[\Gamma(k+\frac{1}{2})]^2} (u(k) - 4v(k)) \right] \\
&= I^2(P) \frac{-k^2 + 2k - 1 + nk - n}{n(k-1)(k-2)} \\
&\quad + I^2(P) \frac{1}{(3s-2)^2} \left[\frac{-k + 1 + 8k^2n - 23kn + 15n}{2n(k-1)(k-2)} + \frac{3[\Gamma(k-1)]^2}{[\Gamma(k+\frac{1}{2})]^2} (u(k) - 4v(k)) \right] \\
&= I^2(P) \frac{n(k-1) - (k-1)^2}{n(k-1)(k-2)} \\
&\quad + I^2(P) \frac{1}{(3s-2)^2} \left[\frac{-k + 1 + 8k^2n - 24kn + 16n + kn - n}{2n(k-1)(k-2)} + \frac{3[\Gamma(k-1)]^2}{[\Gamma(k+\frac{1}{2})]^2} (u(k) - 4v(k)) \right] \\
&= I^2(P) \frac{n-k+1}{n(k-2)} + I^2(P) \frac{1}{(3s-2)^2} \left[4 + \frac{n(k-1) - (k-1)}{2n(k-1)(k-2)} + \frac{3[\Gamma(k-1)]^2}{[\Gamma(k+\frac{1}{2})]^2} (u(k) - 4v(k)) \right] \\
&= I^2(P) \left(\frac{n-k+1}{n(k-2)} + \frac{1}{(2-3s)^2} \left[4 + \frac{n-1}{2n(k-2)} + \frac{3[\Gamma(k-1)]^2}{[\Gamma(k+\frac{1}{2})]^2} (u(k) - 4v(k)) \right] \right)
\end{aligned} \tag{A.66}$$

A.7. Original photon mapping with cone filter

The variance is:

$$\begin{aligned} \text{Var}[\hat{I}] &= \sum_{i=1}^{k-1} \text{Var}[\hat{I}_i^*] + 2 \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} E[\hat{I}_i^* \hat{I}_j^*] - 2 \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} E[\hat{I}_i^*] E[\hat{I}_j^*] \\ &\quad + \text{Var}[\hat{I}_k^*] + 2 \sum_{i=1}^{k-1} E[\hat{I}_i^* \hat{I}_k^*] - 2 \sum_{i=1}^{k-1} E[\hat{I}_i^*] E[\hat{I}_k^*] \end{aligned} \quad (\text{A.67})$$

$$\text{Var}[\hat{I}_k^*]$$

$$\text{Var}[\hat{I}_k^*] = E[\hat{I}_k^{*2}] - (E[\hat{I}_k^*])^2 \quad (\text{A.68})$$

A.7.0.1. $E[\hat{I}_k^{*2}]$

$$\begin{aligned} E[\hat{I}_k^{*2}] &= \int_0^1 \hat{I}_k^{*2}(r_k) f_{X(k)}(r_k) dr_k \\ &= \int_0^1 \left(\frac{1 - \frac{1}{s}}{\pi r_k^2 (1 - \frac{2}{3s})} \phi \right)^2 \mathcal{C} 2 r_k^{2k-1} (1 - r_k^2)^{n-k} dr_k \\ &= \left(\frac{1 - \frac{1}{s}}{\pi (1 - \frac{2}{3s})} \phi \right)^2 \int_0^1 \frac{1}{r_k^4} \mathcal{C} 2 r_k^{2k-1} (1 - r_k^2)^{n-k} dr_k \\ &= 2\mathcal{C} \left(\frac{1 - \frac{1}{s}}{\pi (1 - \frac{2}{3s})} \phi \right)^2 \int_0^1 r_k^{2k-5} (1 - r_k^2)^{n-k} dr_k \\ &= 2\mathcal{C} \left(\frac{1 - \frac{1}{s}}{\pi (1 - \frac{2}{3s})} \phi \right)^2 \frac{\Gamma(k-2)\Gamma(-k+n+1)}{2\Gamma(n-1)} \\ &= 2 \frac{n!}{(k-1)!(n-k)!} \left(\frac{1 - \frac{1}{s}}{\pi (1 - \frac{2}{3s})} \phi \right)^2 \frac{\Gamma(k-2)\Gamma(-k+n+1)}{2\Gamma(n-1)} \\ &= \frac{9(n-1)n(s-1)^2\phi^2}{\pi^2(k-2)(k-1)(2-3s)^2} \\ &= I^2(P) \frac{9(n-1)(s-1)^2}{(k-2)(k-1)n(2-3s)^2} \end{aligned} \quad (\text{A.69})$$

A.7.0.2. $E[\hat{I}_k^{*2}] - (E[\hat{I}_k^*])^2$

From García et al. [GHUPS14], we already know that:

$$E[\hat{I}_k^*] = \frac{(3s-3)I(P)}{(3s-2)(k-1)} \quad (\text{A.70})$$

So we have now

$$\begin{aligned}
\text{Var}[\hat{I}_k^*] &= E[\hat{I}_k^{*2}] - (E[\hat{I}_k^*])^2 \\
&= I^2(P) \frac{9(n-1)(s-1)^2}{(k-2)(k-1)n(2-3s)^2} - \left(\frac{(3s-3)I(P)}{(3s-2)(k-1)} \right)^2 \\
&= I^2(P) \frac{9(s-1)^2(-k+n+1)}{(k-2)(k-1)^2n(2-3s)^2}
\end{aligned} \tag{A.71}$$

$$\sum_{i=1}^{k-1} E[\hat{I}_i^* \hat{I}_k^*]$$

$$\begin{aligned}
E[\hat{I}_i^* \hat{I}_k^*] &= \int_0^1 \int_0^{r_k} \hat{I}_i^*(r_i, r_k) \hat{I}_k^*(r_k) f_{X_{(i,k)}}(r_i, r_k) dr_i dr_k \\
&= \int_0^1 \int_0^{r_k} \left(\frac{1 - \frac{r_i}{sr_k}}{\pi r_k^2 (1 - \frac{2}{3s})} \phi \right) \left(\frac{1 - \frac{1}{s}}{\pi r_k^2 (1 - \frac{2}{3s})} \phi \right) C_K 2r_i^{2i-1} [r_k^2 - r_i^2]^{k-i-1} dr_i dr_k \\
&= 2 \frac{(1 - \frac{1}{s})\phi^2}{\pi^2 (1 - \frac{2}{3s})^2} C_J \int_0^1 \int_0^{r_k} \frac{1}{r_k^4} \left(1 - \frac{r_i}{sr_k} \right) r_k (1 - r_k^2)^{n-k} 2r_i^{2i-1} [r_k^2 - r_i^2]^{k-i-1} dr_i dr_k \\
&= 4 \frac{(1 - \frac{1}{s})\phi^2}{\pi^2 (1 - \frac{2}{3s})^2} C_J \int_0^1 \frac{(1 - r_k^2)^{n-k}}{r_k^3} \int_0^{r_k} \left(1 - \frac{r_i}{sr_k} \right) r_i^{2i-1} [r_k^2 - r_i^2]^{k-i-1} dr_i dr_k
\end{aligned} \tag{A.72}$$

The value of the inner integral is:

$$\begin{aligned}
&\int_0^{r_k} \left(1 - \frac{r_i}{sr_k} \right) r_i^{2i-1} [r_k^2 - r_i^2]^{k-i-1} dr_i \\
&= \frac{r_k^{2k-2} \Gamma(k-i) (s\Gamma(i)\Gamma(k+\frac{1}{2}) - \Gamma(i+\frac{1}{2})\Gamma(k))}{2s\Gamma(k)\Gamma(k+\frac{1}{2})}
\end{aligned} \tag{A.73}$$

Then the value of the outer integral is:

$$\begin{aligned}
&\int_0^1 \frac{(1 - r_k^2)^{n-k}}{r_k^3} \frac{r_k^{2k-2} \Gamma(k-i) (s\Gamma(i)\Gamma(k+\frac{1}{2}) - \Gamma(i+\frac{1}{2})\Gamma(k))}{2s\Gamma(k)\Gamma(k+\frac{1}{2})} dr_k \\
&= \left(\frac{\Gamma(i)}{k^2 - 3k + 2} - \frac{\Gamma(i+\frac{1}{2})\Gamma(k-2)}{s\Gamma(k+\frac{1}{2})} \right) \frac{\Gamma(k-i)\Gamma(-k+n+1)}{4\Gamma(n-1)}
\end{aligned} \tag{A.74}$$

Then we have

$$\begin{aligned}
E[\hat{I}_i^* \hat{I}_k^*] &= 4 \frac{(1 - \frac{1}{s})\phi^2}{\pi^2 (1 - \frac{2}{3s})^2} C_J \left(\frac{\Gamma(i)}{k^2 - 3k + 2} - \frac{\Gamma(i+\frac{1}{2})\Gamma(k-2)}{s\Gamma(k+\frac{1}{2})} \right) \frac{\Gamma(k-i)\Gamma(-k+n+1)}{4\Gamma(n-1)} \\
&= 4 \frac{(1 - \frac{1}{s})\phi^2}{\pi^2 (1 - \frac{2}{3s})^2} \frac{n!}{(i-1)!(k-i-1)!(n-k)!} \left(\frac{\Gamma(i)}{k^2 - 3k + 2} - \frac{\Gamma(i+\frac{1}{2})\Gamma(k-2)}{s\Gamma(k+\frac{1}{2})} \right) \\
&\quad \times \frac{\Gamma(k-i)\Gamma(-k+n+1)}{4\Gamma(n-1)} \\
&= \frac{9(n-1)n(s-1)\phi^2}{\pi^2(2-3s)^2} \left(\frac{s}{k^2 - 3k + 2} - \frac{\Gamma(i+\frac{1}{2})\Gamma(k-2)}{\Gamma(i)\Gamma(k+\frac{1}{2})} \right)
\end{aligned} \tag{A.75}$$

A. Appendix

Finally we have

$$\begin{aligned}
\sum_{i=1}^{k-1} E[\hat{I}_i^* \hat{I}_k^*] &= \sum_{i=1}^{k-1} \frac{9(n-1)n(s-1)\phi^2}{\pi^2(2-3s)^2} \left(\frac{s}{k^2-3k+2} - \frac{\Gamma(i+\frac{1}{2})\Gamma(k-2)}{\Gamma(i)\Gamma(k+\frac{1}{2})} \right) \\
&= \frac{3(n-1)n(s-1)\phi^2}{\pi^2(k-2)(3s-2)} \\
&= I^2(P) \frac{3(n-1)(s-1)}{(k-2)n(3s-2)}
\end{aligned} \tag{A.76}$$

$$\begin{aligned}
\sum_{i=1}^{k-1} E[\hat{I}_i^*] E[\hat{I}_k^*] &= \sum_{i=1}^{k-1} I(P) \left(\frac{3s}{3s-2} \right) \left(\frac{1}{k-1} - \frac{\Gamma(i+\frac{1}{2})\Gamma(k-1)}{s\Gamma(i)\Gamma(k+\frac{1}{2})} \right) \frac{(3s-3)I(P)}{(3s-2)(k-1)} \\
&= I^2(P) \frac{3(s-1)}{(k-1)(3s-2)}
\end{aligned} \tag{A.77}$$

Variance

The variance value is:

$$\begin{aligned}
Var[\hat{I}] &= Var_{k-1} + Var[\hat{I}_k^*] + 2 \sum_{i=1}^{k-1} E[\hat{I}_i^* \hat{I}_k^*] - 2 \sum_{i=1}^{k-1} E[\hat{I}_i^*] E[\hat{I}_k^*] \\
&= Var_{k-1} + I^2(P) \frac{9(s-1)^2(-k+n+1)}{(k-2)(k-1)^2 n(2-3s)^2} + 2I^2(P) \frac{3(n-1)(s-1)}{(k-2)n(3s-2)} - 2I^2(P) \frac{3(s-1)}{(k-1)(3s-2)} \\
&= Var_{k-1} - I^2(P) \frac{3(s-1)(k-n-1)(k(6s-4)-3s+1)}{(k-2)(k-1)^2 n(2-3s)^2} \\
&= I^2(P) \left(\frac{n-k+1}{n(k-2)} + \frac{1}{(2-3s)^2} \left[4 + \frac{n-1}{2n(k-2)} + \frac{3[\Gamma(k-1)]^2}{[\Gamma(k+\frac{1}{2})]^2} (u(k) - 4v(k)) \right] \right) \\
&\quad - I^2(P) \frac{3(s-1)(k-n-1)(k(6s-4)-3s+1)}{(k-2)(k-1)^2 n(2-3s)^2}
\end{aligned} \tag{A.78}$$

A.8. Gaussian filter

Here we give the formula of the sum of the diagonal of covariance matrix $\sum_{i=1}^{k-1} Var[\hat{I}_i^*]$, that we got from our study.

$$\sum_{i=1}^{k-1} Var[\hat{I}_i^*] = \frac{n\phi^2 (H(k) + e^\beta(R(k) + S(k) + T(k)))}{\pi^2 (e^\beta - 1)^2 (\beta + 2e^{\beta/2} - 2e^\beta)^2 (2-k)(k-1)^2} \tag{A.79}$$

with

$$\begin{aligned}
H(k) = & (e^\beta(e^\beta - 2) - k + 1) * [(\beta^2 (\alpha^2(k-1)(n-1) - (k-2)n) \\
& + e^\beta (2\beta^2(k-2)n - 4\alpha^2(\beta-1)(k-1)(n-1)) + e^{2\beta} (4\alpha^2(k-1)(n-1) - \beta^2(k-2)n) \\
& - 8\alpha^2 e^{\frac{3\beta}{2}}(k-1)(n-1) + 4\alpha^2 e^{\beta/2} \beta(k-1)(n-1)]
\end{aligned} \tag{A.80}$$

$$R(k) = \alpha^2(2e^\beta - \beta - 2e^{\beta/2})^2(k-1)(n-1)u(k) \tag{A.81}$$

$$\begin{aligned}
S(k) = & 2v(k)[\beta^2 (\alpha^2(k-1)(n-1) - (k-2)n) + 2e^\beta (\beta^2(k-2)n - 2\alpha^2(\beta-1)(k-1)(n-1)) \\
& + e^{2\beta} (4\alpha^2(k-1)(n-1) - \beta^2(k-2)n) - 8\alpha^2 e^{\frac{3\beta}{2}}(k-1)(n-1) + 4\alpha^2 e^{\beta/2} \beta(k-1)(n-1)]
\end{aligned} \tag{A.82}$$

and

$$T(k) = e^\beta (e^\beta - 1)^2 \beta^2 n(k-2)w(k) \tag{A.83}$$

where function $u(n), v(n)$ and $w(n)$ with $n \in \mathbb{N}$ defined as:

$$\begin{aligned}
u(0) &= 0 & u(1) &= e^\beta \\
u(2) &= e^\beta + (k-1)(-\beta)^{1-k} \Gamma(k-1, 0, -\beta) \\
u(n) &= \frac{(n-k-2)u(n-3) + (2k-3n+\beta+6)u(n-2) + (3n-k-6-\beta)u(n-1)}{n-2}
\end{aligned} \tag{A.84}$$

Here $\Gamma(k-1, 0, -\beta)$ is the generalized incomplete gamma function $\Gamma(k-1, 0) - \Gamma(k-1, -\beta)$

$$\begin{aligned}
v(0) &= 0 & v(1) &= 1 \\
v(2) &= e^{-\frac{\beta}{2}} \beta 2^{k-1} (-\beta)^{-k} ((k-1)\Gamma(k-1, -\frac{\beta}{2}) - \Gamma(k)) + 1 \\
v(n) &= \frac{2(n-k-2)u(n-3) + (4k-6n+\beta+12)u(n-2) + (6n-2k-12-\beta)u(n-1)}{2(n-2)}
\end{aligned} \tag{A.85}$$

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and

$$\begin{aligned}
w(0) &= 0 & w(1) &= 1 \\
w(2) &= 1 + e^{-\beta} 4^{k-1} (k-1)^2 (-\beta)^{2-2k} \left(\Gamma(k-1) - \Gamma\left(k-1, -\frac{\beta}{2}\right) \right)^2 \\
w(3) &= \frac{1}{4} \left[4 + \frac{1}{4} e^{-\beta} (k-1)^2 (-\beta)^{-2k} (4e^{\beta/2} (-\beta)^k + \beta 2^k (\beta + 2k - 4) \left(\Gamma(k-1) - \Gamma\left(k-1, -\frac{\beta}{2}\right) \right) \right)^2 \\
&\quad + e^{-\beta} 4^k (k-1)^2 (-\beta)^{2-2k} \left(\Gamma(k-1) - \Gamma\left(k-1, -\frac{\beta}{2}\right) \right)^2 \right] \\
w(n) &= \frac{q(n)}{-4(n-3)(n-2)^2(4n-\beta-2k-12)}
\end{aligned} \tag{A.86}$$

with

$$\begin{aligned}
q(n) &= 4(n-k-2)(k-n+3)^2(\beta+2k-4n+8)w(n-4) \\
&\quad + (k-n+2)[\beta^3 + 2\beta^2(3k-6n+16) + 4\beta(4k^2 - 13(k+5)n + 35k + 12n^2 + 87) \\
&\quad + 8(2k^3 + k^2(23-9n) + k(n-3)(14n-31) - 8(n-3)^2(n-2))]w(n-3) \\
&\quad - (\beta^2 + 4(k^2 - 3(k+5)n + 7k + 3n^2 + 18) + 4\beta(k-2n+5)) * \\
&\quad (5\beta + 2k^2 + k(\beta - 8n + 22) - 2\beta n + 8(n-5)n + 48) w(n-2) \\
&\quad + (n-3)[- \beta^3 - 8k^3 + 4k^2(-3\beta + 10n - 22) - 2k(-3\beta + 10n - 20)(-\beta + 4n - 10) \\
&\quad + 4\beta^2(3n-7) - 4\beta(n-2)(12n-31) + 64(n-3)(n-2)^2]w(n-1)
\end{aligned} \tag{A.87}$$

A.9. Mathematica code

in figures A.1 to A.6 show our mathematica code with figure A.1: Original PM with constant kernel, figure A.2: Corrected PM with constant kernel, figure A.3 :Epanechnikov kernel, figureA.4: Silverman/Quartic kernel, figure A.5: Corrected PM with cone filter and figure A.6: Original PM with cone filter.

```

We define x := r_i; y := r_j; z := r_k; P := I (P) or PD;

$$\frac{\Gamma[-2+k] \Gamma[1-k+n]}{\Gamma[-1+n]}, \text{Re}[k-n] < 1 \ \&\& \ \text{Re}[k] > 2$$

In[ ]:= c := n! / ((k-1)! * (n-k)!)
In[ ]:= r1(*Integral of Original PM in 2D case*) =
Integrate[ ((z^2)^(k-3)) * ((1-z^2)^(n-k)) * 2 * z, {z, 0, 1} ]
Out[ ]:= ConditionalExpression[  $\frac{\Gamma[-2+k] \Gamma[1-k+n]}{\Gamma[-1+n]}$ , Re[k-n] < 1 && Re[k] > 2]

In[ ]:= v1(*Variance of Original PM in 2D case*) =
FullSimplify[ ((k^2) / (Pi^2)) * ((Pi^2) * (P^2)) / (n^2) *
c *  $\frac{\Gamma[-2+k] \Gamma[1-k+n]}{\Gamma[-1+n]}$  - (((k^2) * (P^2)) / ((k-1)^2)) ]
Out[ ]:=  $\frac{k^2 (1-k+n) P^2}{(-2+k) (-1+k)^2 n}$ 

In[ ]:= r2(*Integral of Original PM in 3D case*) =
Integrate[ (z^(3k-9)) * ((1-z^3)^(n-k)) * 3 * (z^2), {z, 0, 1} ]
Out[ ]:= ConditionalExpression[  $\frac{\Gamma[-2+k] \Gamma[1-k+n]}{\Gamma[-1+n]}$ , Re[k-n] < 1 && Re[k] > 2]

In[ ]:= v2(*Variance of Original PM in 3D case*) =
FullSimplify[ ((9 * (k^2)) / (16 * (Pi^2))) * ((16 * (Pi^2)) * (P^2)) / (9 * (n^2))) *
c *  $\frac{\Gamma[-2+k] \Gamma[1-k+n]}{\Gamma[-1+n]}$  - (((k^2) * (P^2)) / ((k-1)^2)) ]
Out[ ]:=  $\frac{k^2 (1-k+n) P^2}{(-2+k) (-1+k)^2 n}$ 

```

Figure A.1.: Mathematica code of original photon mapping with constant kernel

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```

We define x := r_i; y := r_j; z := r_k; P := I (P) or PD;
|imaginäre Einheitspotenz

In[ ]:= c := n! / ((k - 1)! * (n - k)!)

r1(*Integral of Corrected PM in 2D case*) =
|leite ab
Integrate[ ((z^2)^(k - 3)) * ((1 - z^2)^(n - k)) * 2 * z, {z, 0, 1}]
|integriere

Out[ ]:= ConditionalExpression[ $\frac{\text{Gamma}[-2 + k] \text{Gamma}[1 - k + n]}{\text{Gamma}[-1 + n]}$ , Re[k - n] < 1 && Re[k] > 2]

In[ ]:= v1(*Variance of Corrected PM in 2D case*) =
|Varianz |leite ab
FullSimplify[(((k - 1)^2) / (Pi^2)) * (((Pi^2) * (P^2)) / (n^2)) *
|vereinfache vollständig |Kreiszahl  $\pi$  |Kreiszahl  $\pi$ 
c *  $\frac{\text{Gamma}[-2 + k] \text{Gamma}[1 - k + n]}{\text{Gamma}[-1 + n]} - (P^2)$ ]

Out[ ]:=  $\frac{(1 - k + n) P^2}{(-2 + k) n}$ 

r2(*Integral of Corrected PM in 3D case*) =
|leite ab
Integrate[ (z^(3k - 9)) * ((1 - z^3)^(n - k)) * 3 * (z^2), {z, 0, 1}]
|integriere

Out[ ]:= ConditionalExpression[ $\frac{\text{Gamma}[-2 + k] \text{Gamma}[1 - k + n]}{\text{Gamma}[-1 + n]}$ , Re[k - n] < 1 && Re[k] > 2]

In[ ]:= v2(*Variance of Corrected PM in 3D case*) =
|Varianz |leite ab
FullSimplify[((9 * ((k - 1)^2)) / (16 * (Pi^2))) *
|vereinfache vollständig |Kreiszahl  $\pi$ 
(((16 * (Pi^2)) * (P^2)) / (9 * (n^2))) * c *  $\frac{\text{Gamma}[-2 + k] \text{Gamma}[1 - k + n]}{\text{Gamma}[-1 + n]} - (P^2)$ ]

Out[ ]:=  $\frac{(1 - k + n) P^2}{(-2 + k) n}$ 

```

Figure A.2.: Mathematica code of corrected photon mapping with constant kernel

```

In[ ]:= cj = (n! / (((i - 1)!) * ((k - i - 1)!) * ((n - k)!)))
Out[ ]:= 
$$\frac{n!}{(-1 + i)! (-1 - i + k)! (-k + n)!}$$


p1(*Epanechnikov Inner Integrate E_i^2*) = Integrate[
  ((1 - (x/z)^2)^2 * 2 * (x^((2*i) - 1)) * ((z^2) - (x^2))^(k - i - 1)), {x, 0, z}]
Out[ ]:= ConditionalExpression[
$$\frac{z^{-2+2k} \text{Gamma}[i] \text{Gamma}[2 - i + k]}{\text{Gamma}[2 + k]}$$
,
  Re[i - k] < 2 && Re[z] > 0 && Im[z] == 0 && Re[i] > 0]

p2(*Epanechnikov outer integrate E_i^2*) = Integrate[
  (2 * z * ((1 - (z^2))^(n - k)) * 
$$\frac{z^{-2+2k} \text{Gamma}[i] \text{Gamma}[2 - i + k]}{\text{Gamma}[2 + k]}$$
) / (z^4), {z, 0, 1}]
Out[ ]:= ConditionalExpression[
  (Gamma[i] Gamma[-2 + k] Gamma[2 - i + k] Gamma[1 - k + n]) / (Gamma[2 + k] Gamma[-1 + n]),
  Re[k - n] < 1 && Re[k] > 2]

p3(*Epanechnikov E_i^2*) = FunctionExpand[(((cj * 4 * (phi^2)) / (Pi^2)) *
  ((Gamma[i] Gamma[-2 + k] Gamma[2 - i + k] Gamma[1 - k + n]) / (Gamma[2 + k] Gamma[-1 + n])))
Out[ ]:= 
$$\frac{4 (-1 + i - k) (i - k) (-1 + n) n \phi^2}{(-2 + k) (-1 + k) k (1 + k) \pi^2}$$


p4(*Epanechnikov E_i*) = ((2 * phi) / Pi) * (((k - i) * n) / ((k - 1) * k))
Out[ ]:= 
$$\frac{2 (-i + k) n \phi}{(-1 + k) k \pi}$$


p5(*Epanechnikov Var_i*) = FullSimplify[p3 - (p4^2)]
Out[ ]:= 
$$-\left(4 (i - k) n (-(-1 + k) k (1 - i + k) + (-2 i + k + k^2) n) \phi^2\right) / \left((-2 + k) (-1 + k)^2 k^2 (1 + k) \pi^2\right)$$


p9(*Epanechnikov Sum Var I_i*) = FullSimplify[Sum[p5, {i, 1, (k - 1)}]]
Out[ ]:= 
$$\frac{2 n (-2 n + k (2 - 2 k + 3 n)) \phi^2}{3 (-2 + k) (-1 + k) k \pi^2}$$


r1(*Epanechnikov Sum Ei*Ej*) =
  FullSimplify[Sum[(((2 * phi) / Pi) * (((k - i) * n) / ((k - 1) * k)) *
    ((2 * phi) / Pi) * (((k - j) * n) / ((k - 1) * k))), {j, 1, k - 1}, {i, 1, j - 1}]]
Out[ ]:= 
$$\frac{(-2 + k) (-1 + 3 k) n^2 \phi^2}{6 (-1 + k) k \pi^2}$$


```

Figure A.3.: Mathematica code of corrected photon mapping with epanechnikov kernel

```

t1(*Epanechnikov Inner Integrate by E_ij*) =
  Integrate[(1 - (x/z)^2) * (x^(2+i-1)) * ((y^2-x^2)^(j-i-1)), {x, 0, y}]
  ConditionalExpression[
    
$$\frac{y^{-2+2j} (-iy^2 + jz^2) \Gamma[i] \Gamma[-i+j]}{2z^2 \Gamma[1+j]}$$
,
    Re[i] < Re[j] && Re[y] > 0 && y == Re[y] && Re[i] > 0]

t2(*Epanechnikov Middle Integrate by E_ij*) =
  Integrate[(1 - (y/z)^2) * y * ((z^2-y^2)^(k-j-1)) *
    
$$\frac{y^{-2+2j} (-iy^2 + jz^2) \Gamma[i] \Gamma[-i+j]}{2z^2 \Gamma[1+j]}$$
, {y, 0, z}]
  ConditionalExpression[
    ((1-i+k) z^{-2+2k} \Gamma[i] \Gamma[-i+j] \Gamma[1-j+k]) / (4 \Gamma[2+k]),
    Re[j] < 1 + Re[k] && Re[z] > 0 && z == Re[z] && Re[j] > 0]

t3(*Epanechnikov outer Integrate by E_ij*) = Integrate[(((1-z^2)^(n-k)) / (z^3)) *
  (((1-i+k) z^{-2+2k} \Gamma[i] \Gamma[-i+j] \Gamma[1-j+k]) / (4 \Gamma[2+k])), {z, 0, 1}]
  ConditionalExpression[
    
$$\frac{(1-i+k) \Gamma[i] \Gamma[-i+j] \Gamma[-2+k] \Gamma[1-j+k] \Gamma[1-k+n]}{8 \Gamma[2+k] \Gamma[-1+n]}$$
,
    Re[k-n] < 1 && Re[k] > 2]

t4(*Epanechnikov E_ij*) = FullSimplify[
  ((n! / ((i-1)! * (j-i-1)! * (k-j-1)! * (n-k)!)) * 32 * (\phi^2) / (\pi^2)) *
  
$$\frac{(1-i+k) \Gamma[i] \Gamma[-i+j] \Gamma[-2+k] \Gamma[1-j+k] \Gamma[1-k+n]}{8 \Gamma[2+k] \Gamma[-1+n]}$$

  ]
  ConditionalExpression[
    
$$\frac{4(1-i+k)(-j+k)(-1+n)n\phi^2}{(-2+k)(-1+k)k(1+k)\pi^2}$$


t5(*Epanechnikov Sum E_ij*) = FullSimplify[Sum[r6, {j, 1, k-1}, {i, 1, j-1}]]
  ConditionalExpression[
    
$$\frac{(-1+n)n\phi^2}{2\pi^2}$$


v(*Epanechnikov Var*) = FullSimplify[p9 + 2 * t5 - 2 * r1]
  ConditionalExpression[
    
$$\frac{n(2-3k+4n)\phi^2}{3(-2+k)\pi^2}$$


```

Figure A.3.: Mathematica code of corrected photon mapping with epanechnikov kernel

```

In[ ]:= p1(*Silverman inner Integral E_i^2*) = Integrate[
    |Integriere
    ((1 - ((x / z) ^ 2)) ^ 4) * 2 * (x ^ ((2 * i) - 1)) * ((z ^ 2) - (x ^ 2)) ^ (k - i - 1), {x, 0, z}]
Out[ ]:= ConditionalExpression[
    z^{-2+2k} Gamma[i] Gamma[4 - i + k]
    Gamma[4 + k],
    Re[i - k] < 4 && Re[z] > 0 && Im[z] == 0 && Re[i] > 0]

In[ ]:= p2(*Silverman outer Integral E_i^2*) = Integrate[
    |Integriere
    (2 * z * ((1 - (z ^ 2)) ^ (n - k)) * z^{-2+2k} Gamma[i] Gamma[4 - i + k]) / Gamma[4 + k] / (z ^ 4), {z, 0, 1}]
Out[ ]:= ConditionalExpression[
    Gamma[i] Gamma[-2 + k] Gamma[4 - i + k] Gamma[1 - k + n]
    Gamma[4 + k] Gamma[-1 + n], Re[k - n] < 1 && Re[k] > 2]

In[ ]:= p3(*Silverman E_i^2*) =
    FunctionExpand[(((n! / (((i - 1)!) * ((k - i - 1)!) * ((n - k)!) * 9 * (phi ^ 2)) / (Pi ^ 2)) *
    |multipliziere Funktion aus |Kreiszahl pi
    Gamma[i] Gamma[-2 + k] Gamma[4 - i + k] Gamma[1 - k + n]) /
    Gamma[4 + k] Gamma[-1 + n]]
Out[ ]:=
    9 (-3 + i - k) (-2 + i - k) (-1 + i - k) (i - k) (-1 + n) n phi^2
    (-2 + k) (-1 + k) k (1 + k) (2 + k) (3 + k) pi^2

p4(*Silverman E_i*) = (3 * phi * (k - i + 1) * (k - i) * n) / (Pi * ((k ^ 2) - 1) * k)
    |Kreiszahl pi
Out[ ]:=
    3 (-i + k) (1 - i + k) n phi
    k (-1 + k^2) pi

In[ ]:= p5(*Silverman Var_i*) = FullSimplify[p3 - (p4^2)]
    |vereinfache vollständig
Out[ ]:=
    1
    k^2 pi^2 9 (i - k) n
    (
    ((-3 + i - k) (-2 + i - k) (-1 + i - k) k (-1 + n)) / ((-2 + k) (-1 + k) (1 + k) (2 + k) (3 + k)) -
    (i - k) (1 - i + k)^2 n
    (-1 + k^2)^2 ) phi^2

In[ ]:= p6(*Silverman Sum Var_i*) = FullSimplify[Sum[p5, {i, 1, (k - 1)}]]
    |summiere |vereinfache voll...|summiere
Out[ ]:=
    3 n (4 n + k (-3 + 3 k (k - 2 n) + n)) phi^2
    5 (-2 + k) (-1 + k) k (1 + k) pi^2

```

Figure A.4.: Mathematica code of corrected photon mapping with silverman kernel

A. Appendix

2 | Silverman.nb

```

In[ ]:= r1(*Silverman inner Integrate by E_ij*) =
      Integrate[ ((1 - (x/z)^2)^2) * (x^(2*i-1)) * ((y^2-x^2)^(j-i-1)), {x, 0, y}]
Out[ ]:= ConditionalExpression[
      (y^-2+2*j (i (1+i) y^4 - 2 i (1+j) y^2 z^2 + j (1+j) z^4) Gamma[i] Gamma[-i+j]) /
      (2 z^4 Gamma[2+j]), Re[i] < Re[j] && Re[y] > 0 && y == Re[y] && Re[i] > 0]

In[ ]:= r2(*Silverman Middle Integrate by E_ij*) =
      Integrate[ ((1 - (y/z)^2)^2) * y * ((z^2-y^2)^(k-j-1)) *
      ((y^-2+2*j (i (1+i) y^4 - 2 i (1+j) y^2 z^2 + j (1+j) z^4) Gamma[i] Gamma[-i+j]) /
      (2 z^4 Gamma[2+j])), {y, 0, z}]
Out[ ]:= ConditionalExpression[
      ((-3+i-k) (-2+i-k) z^-2+2*k Gamma[i] Gamma[-i+j] Gamma[2-j+k]) / (4 Gamma[4+k]),
      Re[j] < Re[k] && Re[z] > 0 && z == Re[z] && Re[j] > 0]

In[ ]:= r3(*Silverman Outer Integrate by E_ij*) =
      Integrate[ (((1-z^2)^(n-k)) / (z^3)) * (((-3+i-k) (-2+i-k) z^-2+2*k)
      Gamma[i] Gamma[-i+j] Gamma[2-j+k]) / (4 Gamma[4+k]), {z, 0, 1}]
Out[ ]:= ConditionalExpression[
      ((-3+i-k) (-2+i-k) Gamma[i] Gamma[-i+j] Gamma[-2+k] Gamma[2-j+k]
      Gamma[1-k+n]) / (8 Gamma[4+k] Gamma[-1+n]), Re[k-n] < 1 && Re[k] > 2]

In[ ]:= r4(*Silverman E_ij*) = FullSimplify[
      ((n! / ((i-1)! * (j-i-1)! * (k-j-1)! * (n-k)!)) * 72 * (phi^2) / (Pi^2)) *
      (((-3+i-k) (-2+i-k) Gamma[i] Gamma[-i+j] Gamma[-2+k]
      Gamma[2-j+k] Gamma[1-k+n]) / (8 Gamma[4+k] Gamma[-1+n]))]
Out[ ]:= (9 (-3+i-k) (-2+i-k) phi^2 n! Gamma[-2+k] Gamma[2-j+k]) /
      (pi^2 Gamma[4+k] Gamma[-j+k] Gamma[-1+n])

In[ ]:= r5(*Silverman Sum E_ij*) = FullSimplify[Sum[r7, {j, 1, k-1}, {i, 1, j-1}]]
Out[ ]:= (-1+n) n phi^2 / (2 pi^2)

In[ ]:= t1(*Silverman E_j*) = (3 * phi * (k-j+1) * (k-j) * n) / (Pi * ((k^2)-1) * k)
Out[ ]:= 3 (-j+k) (1-j+k) n phi / (k (-1+k^2) pi)

```

Figure A.4.: Mathematica code of corrected photon mapping with silverman kernel

```

In[ ]:= t2(*Silverman Sum Ei*Ej*) = FullSimplify[Sum[p4 * t1, {j, 1, k - 1}, {i, 1, j - 1}]]
      |summiere      |vereinfache voll...|summiere
Out[ ]:= 
$$\frac{(-2 + k) (-3 + k + 5 k^2) n^2 \phi^2}{10 k (-1 + k^2) \pi^2}$$


In[ ]:= v(*Silverman Var*) = FullSimplify[p6 + 2 * r5 - 2 * t2]
      |vereinfache vollständig
Out[ ]:= 
$$\frac{n (1 - 5 k + 9 n) \phi^2}{5 (-2 + k) \pi^2}$$


```

Figure A.4.: Mathematica code of corrected photon mapping with silverman kernel

A. Appendix

```

In[ ]:= r1(*Cone filter inner Integrate by E_i^2*) =
  Integrate[ ((1 - (x / (s * z)))^2) * (x^(2 * i - 1)) * ((z^2 - x^2)^(k - i - 1)), {x, 0, z} ]
Out[ ]:= ConditionalExpression[
  
$$\frac{z^{-2+2k} \left( -\frac{2s \Gamma\left[\frac{1}{2}+i\right]}{\Gamma\left[\frac{1}{2}+k\right]} + \frac{(i+k s^2) \Gamma[i]}{\Gamma[1+k]} \right) \Gamma[-i+k]}{2 s^2},$$

  Re[i] < Re[k] && Re[z] > 0 && z == Re[z] && Re[i] > 0 ]

In[ ]:= r2(*Cone filter Outer Integrate by E_i^2*) = Integrate[ (((1 - z^2)^(n - k)) / (z^3)) *
  z^{-2+2k} \left( -\frac{2s \Gamma\left[\frac{1}{2}+i\right]}{\Gamma\left[\frac{1}{2}+k\right]} + \frac{(i+k s^2) \Gamma[i]}{\Gamma[1+k]} \right) \Gamma[-i+k]
  ]
Out[ ]:= ConditionalExpression[
  
$$\frac{\Gamma[-2+k] \left( -\frac{2s \Gamma\left[\frac{1}{2}+i\right]}{\Gamma\left[\frac{1}{2}+k\right]} + \frac{(i+k s^2) \Gamma[i]}{\Gamma[1+k]} \right) \Gamma[-i+k] \Gamma[1-k+n]}{4 s^2 \Gamma[-1+n]},$$

  Re[k - n] < 1 && Re[k] > 2 ]

In[ ]:= r3(*Cone filter E_i^2*) = FullSimplify[
  4 * ((phi / (Pi * (1 - (2 / (3 * s))))))^2 * (n! / ((i - 1)! * (k - i - 1)! * (n - k)!)) *
  Gamma[-2+k] \left( -\frac{2s \Gamma\left[\frac{1}{2}+i\right]}{\Gamma\left[\frac{1}{2}+k\right]} + \frac{(i+k s^2) \Gamma[i]}{\Gamma[1+k]} \right) \Gamma[-i+k] \Gamma[1-k+n]
  ]
Out[ ]:= 
$$\frac{9 (-1+n) n \phi^2 \left( \frac{i+k s^2}{2k-3k^2+k^3} - \frac{2s \Gamma\left[\frac{1}{2}+i\right] \Gamma[-2+k]}{\Gamma[i] \Gamma\left[\frac{1}{2}+k\right]} \right)}{\pi^2 (2-3s)^2}$$


In[ ]:= r4(*Cone filter E_i*) = (phi / (Pi * (1 - (2 / (3 * s)))))) * n *
  ((1 / (k - 1)) - ((Gamma[i + (1/2)] * Gamma[k - 1]) / (s * Gamma[i] * Gamma[k + (1/2)])))
Out[ ]:= 
$$\frac{n \phi \left( \frac{1}{-1+k} - \frac{\Gamma\left[\frac{1}{2}+i\right] \Gamma[-1+k]}{s \Gamma[i] \Gamma\left[\frac{1}{2}+k\right]} \right)}{\pi \left( 1 - \frac{2}{3s} \right)}$$


In[ ]:= r5(*Cone filter Var_i*) = FullSimplify[r3 - (r4^2)]
Out[ ]:= 
$$\frac{9 n \phi^2 \left( (-1+n) \left( \frac{i+k s^2}{2k-3k^2+k^3} - \frac{2s \Gamma\left[\frac{1}{2}+i\right] \Gamma[-2+k]}{\Gamma[i] \Gamma\left[\frac{1}{2}+k\right]} \right) - n \left( \frac{s}{-1+k} - \frac{\Gamma\left[\frac{1}{2}+i\right] \Gamma[-1+k]}{\Gamma[i] \Gamma\left[\frac{1}{2}+k\right]} \right)^2 \right)}{\pi^2 (2-3s)^2}$$


```

Figure A.5.: Mathematica code of corrected photon mapping with cone filter

2 | Cone Filter new.nb

```

In[ ]:= r6(*Cone filter Sum Var_i*) = FullSimplify[Sum[r5, {i, 1, (k - 1)}]]
      |Kegel      |summiere      |vereinfache voll...|summiere
Out[ ]:= 
$$\frac{1}{2 \pi^2 (2 - 3 s)^2}$$


$$3 n \phi^2 \left( \frac{3 (-1 + k) (-1 + n) + 8 (-1 + k - n) s + 6 (1 - k + n) s^2}{(-2 + k) (-1 + k)} - \frac{1}{\Gamma\left[\frac{1}{2} + k\right]^2} 6 n \Gamma[-1 + k]^2 \right.$$


$$\left. \text{DifferenceRoot}\left[\text{Function}\left[\{\dot{y}, \hat{n}\}, \left\{\left(1 + 2 \hat{n}\right)^2 \dot{y}[\hat{n}] + (-1 - 4 \hat{n} - 8 \hat{n}^2) \dot{y}[1 + \hat{n}] + \right.\right.\right.\right.$$


$$\left.\left.\left. 4 \hat{n}^2 \dot{y}[2 + \hat{n}] = 0, \dot{y}[1] = 0, \dot{y}[2] = \frac{\pi}{4}\right]\right][k]\right)$$


In[ ]:= u1(*Cone filter Inner Integrate by E_ij*) =
      |Kegel      |veral...|integriere
      Integrate[(1 - (x / (s * z))) * (x^(2 * i - 1)) * ((y^2 - x^2)^(j - i - 1)), {x, 0, y}]
      |integriere
Out[ ]:= ConditionalExpression[
      
$$\frac{y^{-2+2j} \left(-y \Gamma\left[\frac{1}{2} + i\right] \Gamma[j] + s z \Gamma[i] \Gamma\left[\frac{1}{2} + j\right]\right) \Gamma[-i + j]}{2 s z \Gamma[j] \Gamma\left[\frac{1}{2} + j\right]},$$

      Re[i] < Re[j] && Re[y] > 0 && y == Re[y] && Re[i] > 0]

In[ ]:= u2(*Middle Integrate by E_ij*) =
      |integriere
      FullSimplify[Integrate[(1 - (y / (s * z))) * y * ((z^2 - y^2)^(k - j - 1)) *
      |vereinfache voll...|integriere
      
$$\frac{y^{-2+2j} \left(-y \Gamma\left[\frac{1}{2} + i\right] \Gamma[j] + s z \Gamma[i] \Gamma\left[\frac{1}{2} + j\right]\right) \Gamma[-i + j]}{2 s z \Gamma[j] \Gamma\left[\frac{1}{2} + j\right]}, \{y, 0, z\}]$$

Out[ ]:= ConditionalExpression[
      
$$\left(z^{-2+2k} \Gamma[-i + j] \left(\Gamma\left[\frac{1}{2} + i\right] \Gamma[j] \Gamma[1 + j] \Gamma[k] \Gamma\left[\frac{1}{2} + k\right] + \right.\right.$$


$$s \Gamma\left[\frac{1}{2} + j\right] \left(-\left(\Gamma\left[\frac{1}{2} + i\right] \Gamma[j] + \Gamma[i] \Gamma\left[\frac{1}{2} + j\right]\right) \Gamma[k] + \right.$$


$$\left.\left. s \Gamma[i] \Gamma[j] \Gamma\left[\frac{1}{2} + k\right]\right) \Gamma[1 + k]\right) \Gamma[-j + k] \Big/$$


$$\left(4 s^2 \Gamma[j] \Gamma\left[\frac{1}{2} + j\right] \Gamma[k] \Gamma\left[\frac{1}{2} + k\right] \Gamma[1 + k]\right),$$

      Re[j] < Re[k] && Re[z] > 0 && z == Re[z] && Re[j] > 0]

```

Figure A.5.: Mathematica code of corrected photon mapping with cone filter

```

In[ ]:= u3(*Cone filter Outer Integrate by E_ij*) =
  |Kegel |äuße...|Integriere
  Integrate[(((1 - z^2)^(n - k)) / (z^3)) * (z^-2+2k Gamma[-i + j]
  |Integriere |Gammafunktion
  (s Gamma[i] Gamma[1/2 + j] (-Gamma[1/2 + j] Gamma[k] + s Gamma[j] Gamma[1/2 + k])
  |Gammafunkt... |Gammafunktion |Gammafunkt... |Gammafunkt... |Gammafunkt... |Gammafunkt...
  Gamma[1 + k] + Gamma[1/2 + i] Gamma[j] Gamma[k]
  |Gammafunkt... |Gammafunkt... |Gammafunkt... |Gammafunkt...
  (Gamma[1 + j] Gamma[1/2 + k] - s Gamma[1/2 + j] Gamma[1 + k]) Gamma[-j + k]) /
  |Gammafunkt... |Gammafunkt... |Gammafunkt... |Gammafunkt... |Gammafunkt... |Gammafunkt...
  (4 s^2 Gamma[j] Gamma[1/2 + j] Gamma[k] Gamma[1/2 + k] Gamma[1 + k]), {z, 0, 1}]

Out[ ]:= ConditionalExpression[
  (Gamma[-i + j] Gamma[-2 + k] (s Gamma[i] Gamma[1/2 + j] (-Gamma[1/2 + j] Gamma[k] +
  s Gamma[j] Gamma[1/2 + k]) Gamma[1 + k] + Gamma[1/2 + i] Gamma[j]
  Gamma[k] (Gamma[1 + j] Gamma[1/2 + k] - s Gamma[1/2 + j] Gamma[1 + k]))
  Gamma[-j + k] Gamma[1 - k + n]) / (8 s^2 Gamma[j] Gamma[1/2 + j] Gamma[k]
  Gamma[1/2 + k] Gamma[1 + k] Gamma[-1 + n]), Re[k - n] < 1 && Re[k] > 2]
  
```

Figure A.5.: Mathematica code of corrected photon mapping with cone filter

4 | Cone Filter new.nb

```

In[ ]:= u4(*Cone filter E_ij*) =
  FullSimplify[ ((n! / ((i - 1)! * (j - i - 1)! * (k - j - 1)! * (n - k)!)) * 8 * (phi^2)) /
    ((Pi^2) * ((1 - (2 / (3 * s)))^2)) *
    (Gamma[-i + j] Gamma[-2 + k] (s Gamma[i] Gamma[1/2 + j]
      (-Gamma[1/2 + j] Gamma[k] + s Gamma[j] Gamma[1/2 + k]) Gamma[1 + k] + Gamma[1/2 + i]
      Gamma[j] Gamma[k] (Gamma[1 + j] Gamma[1/2 + k] - s Gamma[1/2 + j] Gamma[1 + k]))
      Gamma[-j + k] Gamma[1 - k + n]) / (8 s^2 Gamma[j] Gamma[1/2 + j] Gamma[k]
      Gamma[1/2 + k] Gamma[1 + k] Gamma[-1 + n])
    Out[ ]:= (9 * 2^2 (-1 + j + k) phi^2 n! Gamma[-2 + k] (Gamma[1/2 + i] Gamma[j] Gamma[1 + j] Gamma[k] Gamma[1/2 + k] +
      s Gamma[1/2 + j] (-Gamma[1/2 + i] Gamma[j] + Gamma[i] Gamma[1/2 + j]) Gamma[k] +
      s Gamma[i] Gamma[j] Gamma[1/2 + k]) Gamma[1 + k]) /
      (pi^3 (2 - 3 s)^2 Gamma[i] Gamma[2 j] Gamma[2 k] Gamma[1 + k] Gamma[-1 + n])

In[ ]:= u5(*Cone filter Sum E_ij*) = FullSimplify[Sum[u4, {j, 1, k - 1}, {i, 1, j - 1}]]
  Out[ ]:= ((-1 + n) n phi^2) / (2 pi^2)

In[ ]:= t1(*Cone filter E_j*) = (phi / (Pi * (1 - (2 / (3 * s))))) * n *
  ((1 / (k - 1)) - ((Gamma[j + (1/2)] * Gamma[k - 1]) / (s * Gamma[j] * Gamma[k + (1/2)])))
  Out[ ]:= (n phi (1 / (-1 + k) - (Gamma[1/2 + j] Gamma[-1 + k]) / (s Gamma[j] Gamma[1/2 + k]))) /
  (pi (1 - 2 / (3 s)))

In[ ]:= t2(*Cone filter Sum Ei*Ej {i,1,j-1}*) = FullSimplify[Sum[r4 * t1, {i, 1, j - 1}]]
  Out[ ]:= (3 (-1 + j) n^2 phi^2 (-Gamma[1/2 + j] Gamma[k] + s Gamma[j] Gamma[1/2 + k])
    (-2 Gamma[1/2 + j] Gamma[k] + 3 s Gamma[j] Gamma[1/2 + k])) /
    ((-1 + k)^2 pi^2 (2 - 3 s)^2 Gamma[j]^2 Gamma[1/2 + k]^2)

```

Figure A.5.: Mathematica code of corrected photon mapping with cone filter

$$\begin{aligned}
 \text{In[]:= } & \mathbf{t3}(\text{*Cone filter Sum Ei*Ej*}) = \text{FullSimplify}[\text{Sum}[\mathbf{t2}, \{\mathbf{j}, 1, \mathbf{k} - 1\}]] \\
 & \quad \text{[Kegel] [summiere] [vereinfache voll...]} \text{summiere} \\
 \text{Out[]:= } & \frac{1}{2 \pi^2 (2 - 3 s)^2} \\
 & 3 n^2 \phi^2 \left(\frac{(-2 + k) s (-4 + 3 s)}{-1 + k} + \frac{1}{\text{Gamma}[\frac{1}{2} + k]^2} 4 \text{Gamma}[-1 + k]^2 \left(\text{DifferenceRoot}[\text{Function}[\{ \dot{y}, \dot{n} \}, \{ (1 + 2 \dot{n})^2 \dot{y}[\dot{n}] + (-1 - 4 \dot{n} - 8 \dot{n}^2) \dot{y}[1 + \dot{n}] + 4 \dot{n}^2 \dot{y}[2 + \dot{n}] = 0, \dot{y}[1] = 0, \dot{y}[2] = \frac{\pi}{4} \}]] [k] + \text{DifferenceRoot}[\text{Function}[\{ \dot{y}, \dot{n} \}, \{ (1 + \dot{n}) (1 + 2 \dot{n})^2 \dot{y}[\dot{n}] + (-1 - 5 \dot{n} - 8 \dot{n}^2 - 8 \dot{n}^3) \dot{y}[1 + \dot{n}] + 4 \dot{n}^3 \dot{y}[2 + \dot{n}] = 0, \dot{y}[1] = 0, \dot{y}[2] = \frac{\pi}{4} \}]] [k] \right) \right) \\
 \\
 \text{In[]:= } & \mathbf{v}(\text{*Cone filter Corrected PM Var*}) = \text{FullSimplify}[\mathbf{r6} + 2 * \mathbf{u5} - 2 * \mathbf{t3}] \\
 & \quad \text{[Kegel] [vereinfache vollständig]} \\
 \text{Out[]:= } & \frac{1}{2 \pi^2 (2 - 3 s)^2} \\
 & n \phi^2 \left(\frac{(-7 + 8 k) (-1 + n) + 24 (-1 + k - n) s + 18 (1 - k + n) s^2}{-2 + k} + \frac{1}{\text{Gamma}[\frac{1}{2} + k]^2} 6 n \text{Gamma}[-1 + k]^2 \left(\text{DifferenceRoot}[\text{Function}[\{ \dot{y}, \dot{n} \}, \{ (1 + 2 \dot{n})^2 \dot{y}[\dot{n}] + (-1 - 4 \dot{n} - 8 \dot{n}^2) \dot{y}[1 + \dot{n}] + 4 \dot{n}^2 \dot{y}[2 + \dot{n}] = 0, \dot{y}[1] = 0, \dot{y}[2] = \frac{\pi}{4} \}]] [k] - 4 \text{DifferenceRoot}[\text{Function}[\{ \dot{y}, \dot{n} \}, \{ (1 + \dot{n}) (1 + 2 \dot{n})^2 \dot{y}[\dot{n}] + (-1 - 5 \dot{n} - 8 \dot{n}^2 - 8 \dot{n}^3) \dot{y}[1 + \dot{n}] + 4 \dot{n}^3 \dot{y}[2 + \dot{n}] = 0, \dot{y}[1] = 0, \dot{y}[2] = \frac{\pi}{4} \}]] [k] \right) \right) \\
 \\
 \text{In[]:= } & \mathbf{s} = 1 \\
 \text{Out[]:= } & 1 \\
 \\
 \text{In[]:= } & \mathbf{vt}(\text{*Triangle Var*}) = \text{FullSimplify}[\mathbf{v}] \\
 & \quad \text{[Dreieck] [vereinfache vollständig]} \\
 \text{Out[]:= } & \frac{1}{2 \pi^2} n \phi^2 \left(\frac{1 - 13 n + k (-2 + 8 n)}{-2 + k} + \frac{1}{\text{Gamma}[\frac{1}{2} + k]^2} 6 n \text{Gamma}[-1 + k]^2 \left(\text{DifferenceRoot}[\text{Function}[\{ \dot{y}, \dot{n} \}, \{ (1 + 2 \dot{n})^2 \dot{y}[\dot{n}] + (-1 - 4 \dot{n} - 8 \dot{n}^2) \dot{y}[1 + \dot{n}] + 4 \dot{n}^2 \dot{y}[2 + \dot{n}] = 0, \dot{y}[1] = 0, \dot{y}[2] = \frac{\pi}{4} \}]] [k] - 4 \text{DifferenceRoot}[\text{Function}[\{ \dot{y}, \dot{n} \}, \{ (1 + \dot{n}) (1 + 2 \dot{n})^2 \dot{y}[\dot{n}] + (-1 - 5 \dot{n} - 8 \dot{n}^2 - 8 \dot{n}^3) \dot{y}[1 + \dot{n}] + 4 \dot{n}^3 \dot{y}[2 + \dot{n}] = 0, \dot{y}[1] = 0, \dot{y}[2] = \frac{\pi}{4} \}]] [k] \right) \right)
 \end{aligned}$$

Figure A.5.: Mathematica code of corrected photon mapping with cone filter

```

In[4]:= r1(*Cone filter Integrate by E_k^2*) =
      |Kegel |integriere
      Integrate[(z^(2*k-5)) * ((1 - (z^2))^(n-k)), {z, 0, 1}]
      |integriere
Out[4]= ConditionalExpression[ $\frac{\text{Gamma}[-2+k] \text{Gamma}[1-k+n]}{2 \text{Gamma}[-1+n]}$ , Re[k-n] < 1 && Re[k] > 2]

In[5]:= r2(*Cone filter E_k^2*) = FullSimplify[2 * (n! / ((k-1)! * (n-k)!)) *
      |Kegel |vereinfache vollständig
      (((1 - (1/s)) * phi) / (Pi * (1 - (2/(3*s))))))^2 *  $\frac{\text{Gamma}[-2+k] \text{Gamma}[1-k+n]}{2 \text{Gamma}[-1+n]}$ ]
      |Kreiszahl pi
Out[5]=  $\frac{9(-1+n)n(-1+s)^2\phi^2}{(-2+k)(-1+k)\pi^2(2-3s)^2}$ 

In[9]:= r3(*Cone filter E_k*) = ((3*s-3) * (phi*n) / Pi) / ((3*s-2) * (k-1))
      |Kegel |Kreiszahl pi
Out[9]=  $\frac{n(-3+3s)\phi}{(-1+k)\pi(-2+3s)}$ 

In[12]:= r4(*Cone filter Var_k*) = FullSimplify[r2 - (r3^2)]
      |Kegel |vereinfache vollständig
Out[12]=  $\frac{9n(1-k+n)(-1+s)^2\phi^2}{(-2+k)(-1+k)^2\pi^2(2-3s)^2}$ 

In[11]:= u1(*Cone filter Inner Integrate by E_ik*) =
      |Kegel |verat... |integriere
      Integrate[(1 - (x / (s*z))) * (x^(2*i-1)) * ((z^2-x^2)^(k-i-1)), {x, 0, z}]
      |integriere
Out[11]= ConditionalExpression[
      (z^-2+2k (-Gamma[1/2+i] Gamma[k] + s Gamma[i] Gamma[1/2+k]) Gamma[-i+k]) /
      (2 s Gamma[k] Gamma[1/2+k]), Re[i] < Re[k] && Re[z] > 0 && z == Re[z] && Re[i] > 0]

In[13]:= u2(*Cone filter Outer Integrate by E_ik*) =
      |Kegel |äuße... |integriere
      Integrate[(((1 - (z^2))^(n-k)) / (z^3)) *
      ((z^-2+2k (-Gamma[1/2+i] Gamma[k] + s Gamma[i] Gamma[1/2+k]) Gamma[-i+k]) /
      (2 s Gamma[k] Gamma[1/2+k])), {z, 0, 1}]
      |Gammafunktion |Gammafunkt... |Gammaf... |Gammafunktion |Gammafunktion
Out[13]= ConditionalExpression[
      ((Gamma[i] / (2-3k+k^2) - Gamma[1/2+i] Gamma[-2+k] / (s Gamma[1/2+k])) Gamma[-i+k] Gamma[1-k+n]) /
      (4 Gamma[-1+n]), Re[k-n] < 1 && Re[k] > 2]

```

Figure A.6.: Mathematica code of original photon mapping with cone filter

A. Appendix

2 | Cone Filter PM.nb

```

In[15]:= u3(*Cone filter E_ik*) =
  |Kegel
  FullSimplify[ ((n! / ((i - 1)! * (k - i - 1)! * (n - k)!)) * 4 * (phi^2) * (1 - (1/s))) /
  |vereinfache vollständig
  ((Pi^2) * ((1 - (2 / (3 * s)))) ^ 2) *
  |Kreiszahl pi
  (((Gamma[i] / (2 - 3 k + k^2) - (Gamma[1/2 + i] Gamma[-2 + k]) / (s Gamma[1/2 + k]))
  Gamma[-i + k] Gamma[1 - k + n]) / (4 Gamma[-1 + n]))
  |Gammafunktion |Gammafunktion |Gammafunktion

Out[15]= 9 (-1 + n) n (-1 + s) phi^2 (s / (2 - 3 k + k^2) - Gamma[1/2 + i] Gamma[-2 + k] / (Gamma[i] Gamma[1/2 + k])) / (pi^2 (2 - 3 s)^2)

In[19]:= u4(*Cone filter Sum E_ik*) = FullSimplify[Sum[u3, {i, 1, k - 1}]]
  |Kegel |summiere |vereinfache voll... |summiere

Out[19]= 3 (-1 + n) n (-1 + s) phi^2 / ((-2 + k) pi^2 (-2 + 3 s))

In[22]:= t1(*Cone filter E_i*) = (phi / (Pi * (1 - (2 / (3 * s))))) * n *
  |Kegel |Kreiszahl pi
  ((1 / (k - 1)) - ((Gamma[i + (1/2)] * Gamma[k - 1]) / (s * Gamma[i] * Gamma[k + (1/2)])))
  |Gammafunktion |Gammafunktion |Gammafunktion |Gammafunktion

Out[22]= n phi (1 / (-1 + k) - Gamma[1/2 + i] Gamma[-1 + k] / (s Gamma[i] Gamma[1/2 + k])) / (pi (1 - 2 / (3 s)))

In[25]:= t2(*Cone filter Sum Ei*Ek*) = FullSimplify[Sum[r3 * t1, {i, 1, k - 1}]]
  |Kegel |summiere |vereinfache voll... |summiere

Out[25]= 3 n^2 (-1 + s) phi^2 / ((-1 + k) pi^2 (-2 + 3 s))

In[26]:= v1(*Cone filter Var_k*) = FullSimplify[r4 + 2 * u4 - 2 * t2]
  |Kegel |vereinfache vollständig

Out[26]= - ((3 (-1 + k - n) n (-1 + s) (1 - 3 s + k (-4 + 6 s)) phi^2) / ((-2 + k) (-1 + k)^2 pi^2 (2 - 3 s)^2))

```

Figure A.6.: Mathematica code of original photon mapping with cone filter

```

In[27]:= v2(*Cone filter Corrected PM Var*) =
  |Kegel

$$\frac{1}{2 \pi^2 (2 - 3 s)^2} n \phi^2 \left( \frac{1}{-2 + k} \left( (-7 + 8 k) (-1 + n) + 24 (-1 + k - n) s + 18 (1 - k + n) s^2 \right) + \right.$$


$$\frac{1}{\text{Gamma}\left[\frac{1}{2} + k\right]^2} 6 n \text{Gamma}[-1 + k]^2 \left( \text{DifferenceRoot}\left[\text{Function}\left[\{\dot{y}, \dot{n}\}, \{(1 + 2 \dot{n})^2 \dot{y}[\dot{n}] + \right. \right.\right.$$


$$\left. \left. (-1 - 4 \dot{n} - 8 \dot{n}^2) \dot{y}[1 + \dot{n}] + 4 \dot{n}^2 \dot{y}[2 + \dot{n}] = 0, \dot{y}[1] = 0, \dot{y}[2] = \frac{\pi}{4}\right]\right][k] -$$


$$4 \text{DifferenceRoot}\left[\text{Function}\left[\{\dot{y}, \dot{n}\}, \{(1 + \dot{n}) (1 + 2 \dot{n})^2 \dot{y}[\dot{n}] + \right. \right.$$


$$\left. \left. (-1 - 5 \dot{n} - 8 \dot{n}^2 - 8 \dot{n}^3) \dot{y}[1 + \dot{n}] + 4 \dot{n}^3 \dot{y}[2 + \dot{n}] = 0, \dot{y}[1] = 0, \dot{y}[2] = \frac{\pi}{4}\right]\right][k] \left. \right)$$

Out[27]:=  $\frac{1}{2 \pi^2 (2 - 3 s)^2}$ 

$$n \phi^2 \left( \frac{1}{-2 + k} \left( (-7 + 8 k) (-1 + n) + 24 (-1 + k - n) s + 18 (1 - k + n) s^2 \right) + \frac{1}{\text{Gamma}\left[\frac{1}{2} + k\right]^2}$$


$$6 n \text{Gamma}[-1 + k]^2 \left( \text{DifferenceRoot}\left[\text{Function}\left[\{\dot{y}, \dot{n}\}, \{(1 + 2 \dot{n})^2 \dot{y}[\dot{n}] + \right. \right.\right.$$


$$\left. \left. (-1 - 4 \dot{n} - 8 \dot{n}^2) \dot{y}[1 + \dot{n}] + 4 \dot{n}^2 \dot{y}[2 + \dot{n}] = 0, \dot{y}[1] = 0, \dot{y}[2] = \frac{\pi}{4}\right]\right)[k] -$$


$$4 \text{DifferenceRoot}\left[\text{Function}\left[\{\dot{y}, \dot{n}\}, \{(1 + \dot{n}) (1 + 2 \dot{n})^2 \dot{y}[\dot{n}] + \right. \right.$$


$$\left. \left. (-1 - 5 \dot{n} - 8 \dot{n}^2 - 8 \dot{n}^3) \dot{y}[1 + \dot{n}] + 4 \dot{n}^3 \dot{y}[2 + \dot{n}] = 0, \dot{y}[1] = 0, \dot{y}[2] = \frac{\pi}{4}\right]\right)[k] \left. \right)$$

In[28]:= var(*Cone filter Original PM Var*) = FullSimplify[v1 + v2]
  |Kegel |vereinfache vollständig
Out[28]:=  $\frac{1}{2 \pi^2 (2 - 3 s)^2} n \phi^2$ 

$$\left( \frac{1 - 13 n + k (8 - k - 8 k^2 + 46 n - 23 k n + 8 k^2 n + 12 (1 + 2 k) (-1 + k - n) s + 18 k (1 - k + n) s^2)}{(-2 + k) (-1 + k)^2} + \right.$$


$$\frac{1}{\text{Gamma}\left[\frac{1}{2} + k\right]^2} 6 n \text{Gamma}[-1 + k]^2 \left( \text{DifferenceRoot}\left[\text{Function}\left[\{\dot{y}, \dot{n}\}, \{(1 + 2 \dot{n})^2 \dot{y}[\dot{n}] + \right. \right.\right.$$


$$\left. \left. (-1 - 4 \dot{n} - 8 \dot{n}^2) \dot{y}[1 + \dot{n}] + 4 \dot{n}^2 \dot{y}[2 + \dot{n}] = 0, \dot{y}[1] = 0, \dot{y}[2] = \frac{\pi}{4}\right]\right)[k] -$$


$$4 \text{DifferenceRoot}\left[\text{Function}\left[\{\dot{y}, \dot{n}\}, \{(1 + \dot{n}) (1 + 2 \dot{n})^2 \dot{y}[\dot{n}] + \right. \right.$$


$$\left. \left. (-1 - 5 \dot{n} - 8 \dot{n}^2 - 8 \dot{n}^3) \dot{y}[1 + \dot{n}] + 4 \dot{n}^3 \dot{y}[2 + \dot{n}] = 0, \dot{y}[1] = 0, \dot{y}[2] = \frac{\pi}{4}\right]\right)[k] \left. \right)$$


```

Figure A.6.: Mathematica code of original photon mapping with cone filter

