

INSTITUT FÜR INFORMATIK  
DER LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN



Bachelorarbeit

# Variance of photon mapping with stratification

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Abgabetermin: date



Hiermit versichere ich, dass ich die vorliegende Bachelorarbeit selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel verwendet habe.

.....  
*(Unterschrift des Kandidaten)*



## Abstract

This thesis studies the theoretical aspect of the photon mapping algorithm based on the framework laid out in the paper from Rubén Jesús García-Hernández “Description and Solution of an Unreported Intrinsic Bias in Photon Mapping Density Estimation with Constant Kernel” [GUS12] by applying order statistics to the algorithm to derive theoretical results. We start by recapping the previous results while deriving a first-order approximation to the results. We explain how to reduce the over bias while keeping the variance the same.

Furthermore, to reduce variance, we apply stratification to our model and look at the results. We are building upon the previous works:

Kathrin Hartmann, in her bachelor thesis – “Theoretical study of photon mapping with stratification” [Har20] – calculates the expectation values of the irradiance.

Zhiming Gan, in his bachelor thesis – “Variance of photon mapping density estimation” [Gan18] – calculates the variances for different filtering kernels.

We extend this work by calculating the variance and derive an approximation to expectation values and variance.

In the last part of the work, we provide a proof of concept. We show that our theoretical results are in good agreement with experimental values from our simulation.



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# 1. Introduction

This part will give an overview of the basics of photon mapping, statistics, basic mathematical tools, and the previous work on theoretical photon mapping.

## 1.1. Computer Graphics



Figure 1.1.: (a) Picture of Pong. One of the earlier Video games Original: Bumml3Vector: Beao (<https://commons.wikimedia.org/wiki/File:Pong.svg>), "Pong", marked as public domain, more details on Wikimedia Commons: <https://commons.wikimedia.org/wiki/Template:PD-ineligible> (b) Photorealistic image Gilles Tran ([https://commons.wikimedia.org/wiki/File:Glasses\\_800\\_edit.png](https://commons.wikimedia.org/wiki/File:Glasses_800_edit.png)), "Glasses 800 edit", marked as public domain, more details on Wikimedia Commons: <https://commons.wikimedia.org/wiki/Template:PD-self>

Computer Graphics is a field in computer science centered around generating graphics with the help of algorithms. It plays a significant role in today's movie industry, game industry, prototype visualization, and many other fields.

The topic has made a considerable improvement over its history. From simple pixel images in the early computer games like Pong (2D-pixel) to photo-realistic images or the real-time rendering of realistic-looking video games (see 1.1). Depending on the proposes, unique algorithms are needed. For Computer games, the focus should be more on real-time rendering, while in other applications, the focus is more on realistic rendering. Especially for photo-realistic graphics, there is a need for good algorithms.

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### 1.1.1. Physics of Light

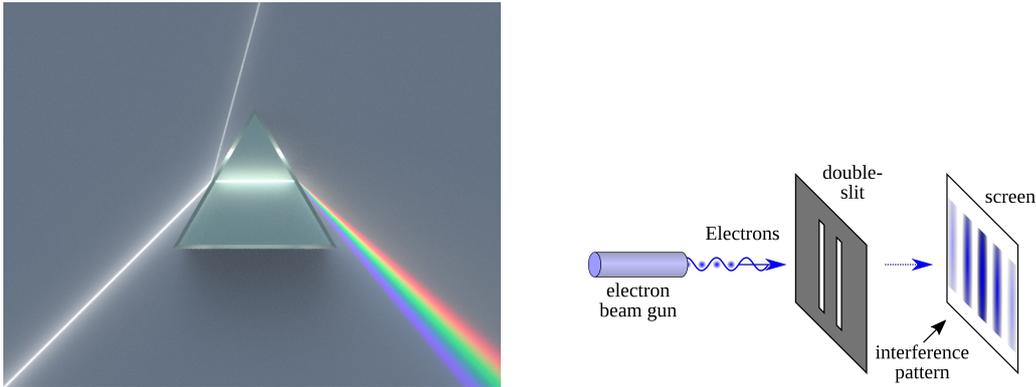


Figure 1.2.: (a) Dispersion of light in a Prism Spigget derivative work: Cepheiden (talk) ([https://commons.wikimedia.org/wiki/File:Dispersive\\_Prism\\_Illustration.jpg](https://commons.wikimedia.org/wiki/File:Dispersive_Prism_Illustration.jpg)), Dispersive Prism Illustration, <https://creativecommons.org/licenses/by-sa/3.0/legalcode> (b) Example setup of double slit experiment Original: NekoJaNekoJaVector: Johannes Kalliauer <https://commons.wikimedia.org/wiki/File:Double-slit.svg>, <https://creativecommons.org/licenses/by-sa/4.0/legalcode>

To better understand those algorithms, we shall look at the physics of light or the different physical models to describe phenomena related to light. How the view of light has changed many times over the history of physics: During the time of Newton, light was viewed as particles with colors. This picture can describe many different phenomena like dispersion (see 1.2 a). The previously mentioned model of light was not complete, but it could describe several characteristics of light. The particle picture can not describe the interference pattern of the double-slit experiment of Thomas Young [You]. The wave picture of the light, on the other hand, can perfectly describe the interference pattern of the double-slit experiment. That duality of light gave rise to many discussions about the "true" nature of light. Only the quantum theory could help solve this puzzle. Light consists of particles called Photons described by a quantum theory. For many purposes, however, it is not necessary to work within a more "complete" picture of light, and one can use a simpler model called geometric optics, which describes light as rays.

### 1.1.2. Photon mapping

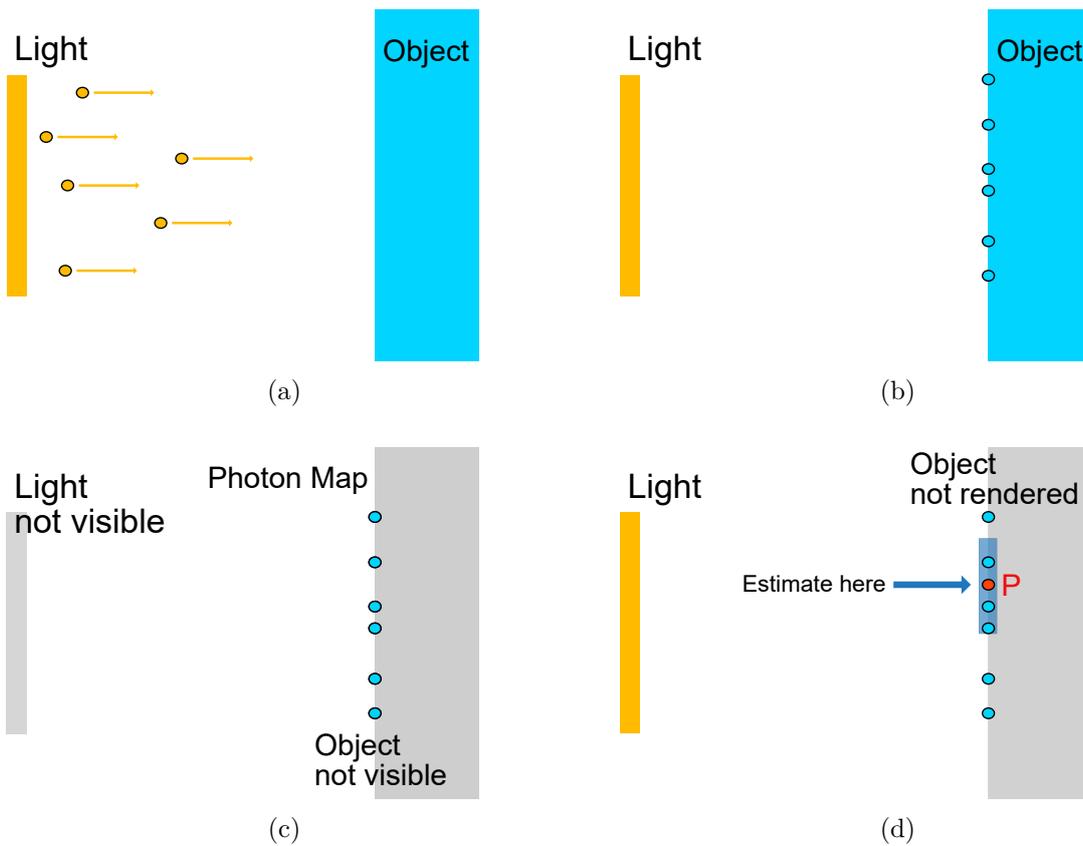


Figure 1.3.: (a) Photons are emitted by a light source (b) Photons hit the object (c) Photons are stored in a data-structure (d) Rendering by performing the estimation at the Point P

Photon mapping is a popular algorithm for creating photo-realistic images. It got developed by Henrik Wann Jensen around 1996 (see [Jen96]). It is able to calculate global illumination including caustics. It is in principle a two-pass algorithm.

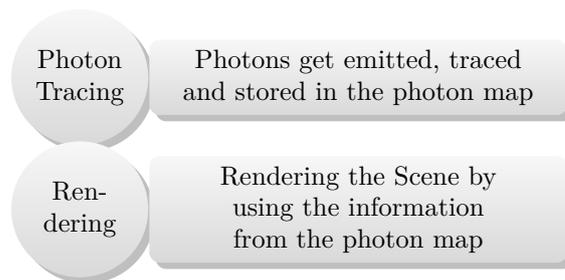


Figure 1.4.: Basic Flowchart for photon mapping. For more details see [Jen01]

The photon mapping is generally stored in a unique data structure – kd-Tree – to improve the performance of the lookup operations for the k nearest photons to query point. We will not go into further detail here but refer to a special book about algorithms like [CLRS09]

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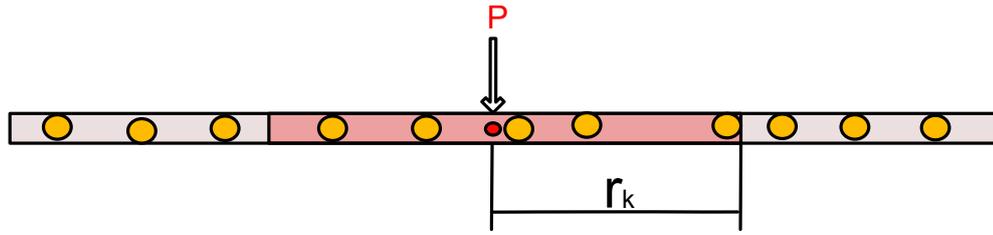


Figure 1.5.: Photo mapping on a line. Collection the  $k$  nearest photons to point  $P$

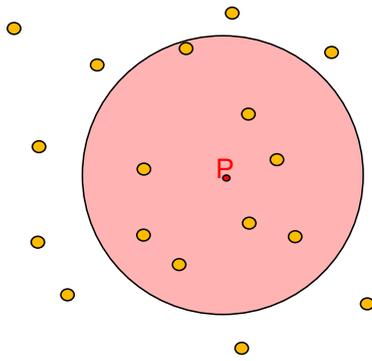


Figure 1.6.: Photo mapping on a surface. Collection the  $k$  nearest photons to point  $P$

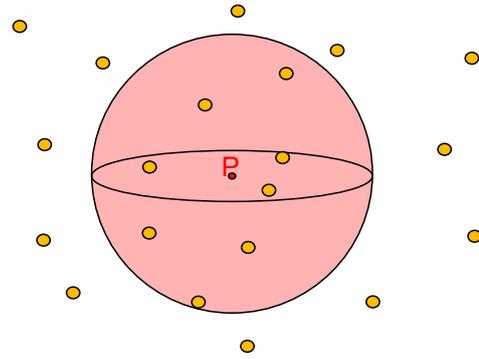


Figure 1.7.: Photo mapping with participating medium. Collecting the  $k$  nearest photons to point  $P$

or in the book from Jensen [Jen01], which describe in detail how to implement the photon mapping algorithm practically.

## 1.2. Statistics

This section gives a brief overview of the basics of statistics, calculus, order statistics, and stratification.

### 1.2.1. Calculus

The integral is a linear operation

$$\int \left( a f(x) + b g(x) \right) dx = a \int f(x) dx + b \int g(x) dx \quad (1.1)$$

and can be seen as the inverse operation of the derivative

$$F = \int f(x) dx = \int \frac{dF(x)}{dx} dx \quad (1.2)$$

$f$  and  $F$  are related as follows:

$$\frac{dF}{dx} = f \quad (1.3)$$

The definitive integral of  $f(x)$  over the interval  $[a, b]$

$$F(b) - F(a) = \int_a^b f(x)dx \quad (1.4)$$

Since we will work in spherical coordinates for most of the time, it is important how integral quantities change by coordinate transformation: For 3D spherical coordinates we have for example:

$$\int_{\sum_i x_i^2 < 1} dx^3 = \int_{\sum_i x_i^2 < 1} r^2 \sin(\theta) dr d\theta d\varphi \quad (1.5)$$

in general the following holds:

$$d^n x = \det(J) d^n q \quad (1.6)$$

Where  $\det(J)$  is the Jacobian-Determinant, which is the n-dimensional volume change from going from cartesian coordinates to general coordinates. Often we will impose normalization conditions like:

$$1 = \int_{S(r \leq 1)^n} f(x) d^n x \quad (1.7)$$

This means the integral over the n-dimensional sphere of some function  $f(x)$  should normalize to 1. Normalization is important for probabilities and probability density function, which follow the normalization to 1 if summed/integrated over their whole domain. For a one dimensional problem with  $r = 1$  the integral will be 2, for a two dimensional problem with  $r = 1$  the integral will be  $\pi$ , and for a two dimensional problem with  $r = 1$  the integral will be  $\frac{4}{3}\pi$ .

$$\text{Ln}(e^x) = x \quad (1.8)$$

and not in base 2, which is quite common in computer science. The logarithm has the important property:

$$\text{Ln}(a) - \text{Ln}(b) = \text{Ln}\left(\frac{a}{b}\right) \quad (1.9)$$

We will make use of some approximations for the final result. In general we can approximate any differential function within some boundaries to a good degree as:

$$f(x) \sim f(x_0) + \frac{df(x_0)}{dx} \Delta x \quad (1.10)$$

### 1.2.2. Special functions

For some calculations, later on, we need some special function. We will state here the most important ones. The Gamma-Function is defined as follows:

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx \quad (n \in \mathbb{N}) \quad (1.11)$$

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We will use the Gamma-Function and factorial interchangeable if  $n$  is an integer. Due to the connection to factorial, this function will show up repeatedly in combinatorial factors. For example,  $n$  distinguishable objects can be ordered in  $n! = \Gamma(n + 1)$  ways. Therefore the probability to find one specific combination is  $\frac{1}{n!} = \frac{1}{\Gamma(n+1)}$ . Additional, we define:

$$0! \equiv 1 \quad (1.12)$$

The following equation, defining the Beta-Function, helps us evaluate some more complicated integral and relates it to the Gamma-Function.

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad (1.13)$$

The beta function in terms of faculties can again be interpreted as a combinatorial factor or as inverse of it. There are  $\frac{n!}{(k-n)!k!}$  different ways to select from a set with  $n$  elements exactly  $k$  elements. The last special function we need, is the hypergeometric function.

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!} \quad (1.14)$$

$${}_2\tilde{F}_1(a, b; c; z) = \frac{{}_2F_1(a, b; c; z)}{\Gamma(c)} \quad (1.15)$$

We use it to solve integrals of the form:

$$B(b, c-b) {}_2F_1(a, b; c; z) = \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a} dx \quad (1.16)$$

For a more in detail explanation of the function see for example [Bro99].

### 1.2.3. Basics

This section will give a short review of the essential elements of statistics. For a more in-deep explanation, see one of the standard statistics textbooks like [FHK<sup>+</sup>16]

One of the principal quantities is the probability or the probability density. This section will mainly deal with the probability or probability density of finding a photon at a certain point. The probability of finding a specific photon  $i$  at  $x$  is given by:

$$p_i(x) \equiv f_i(x) \quad (1.17)$$

where  $f_i(x)$  is a function of position  $x$  describing the probability of photon  $i$ . In the following we often just talk about  $f(x)$  ( $f(x) = f_i(x)$ ) since all photons share the same probability distribution. The cumulated probability  $P$ , describes the probability of finding a photon in the interval  $[a, b]$ :

$$P(a < x < b) \equiv F_{[a,b]}(x) \quad (1.18)$$

One of the most important Quantities of Statistics is the expectation Value  $E[I(x)]$ .  $I(x)$  is just some function depending on  $x$ . It can be seen as the average value of  $I(x)$ . We chose the

function  $I(x)$  here because we are interested in the irradiance in the later chapters, which we will denote with a capital – I. If we are interested in the expectation value of some function  $I(x_i)$ , we can calculate it by summing over all the Values  $I_i$  with their respective values. For discrete cases, it is given by:

$$E[I(x)] = \sum_i p_i I(x_i) \equiv \sum_i p_i I_i \tag{1.19}$$

and for continuous values of x:

$$E[I(x)] = \int p(x) I(x) dx \tag{1.20}$$

Another important quantity is the variance. It is an effective measure for how much the values variate.

$$Var[I(x)] = E[[\hat{I}(x) - E[\hat{I}(x)]]^2] = E[\hat{I}(x)^2] - E[\hat{I}(x)]^2 \tag{1.21}$$

See 1.1 for a table of the most important properties of expectation values and variances.

$E[a + bX + cY] = a + bE[X] + cE[Y]$
$E[f(X)] = \sum_i f_i E[X^i]$ with $f(x) = \sum_i f_i x^i$
$Var[X] = E[(X - E[X])^2]$
$Var[X] = E[X^2] - E[X]^2$
$Var[a + bX] = b^2 Var[X]$

Table 1.1.: Properties of expectation values and variance

### 1.2.4. Ordered Statistic

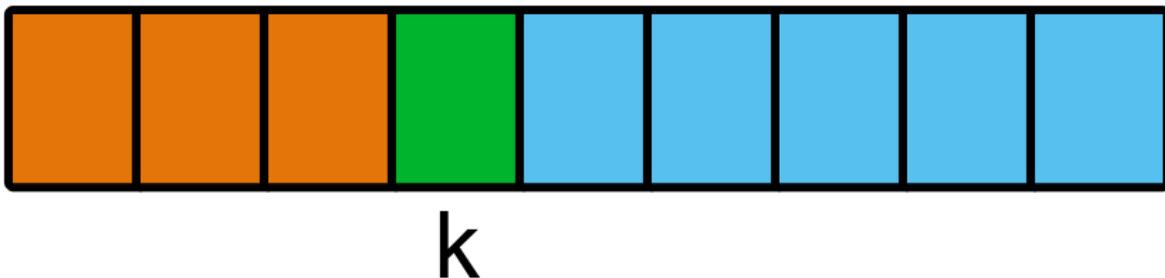


Figure 1.8.: The orange parts are all smaller in value than the green one and the blue ones are all bigger than the green one

Order statistics is the statistic of an ordered list. It can be used to describe the probability of finding an element at a certain position in an ordered list. Since we are for the photon mapping algorithm interested in the k nearest photons, we can use this frame work to

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examine the algorithm further. From the order statistics follows:  $F(x)$ , the accumulated distribution function of  $f(x)$ , is given by:

$$f_k(x) = \frac{n!}{(k-1)!(n-k)!} F^{k-1}(x)(1-F(x))^{n-k} f(x) \quad (1.22)$$

For a more in the deep derivation of the order statistics, see for example [DN04].

Picture 1.8 does illustrate a possible way of thinking about it.  $F^{k-1}(x)$  is the probability to be in the orange part,  $f(x)$  is the probability of being in the green part, and  $(1-F(x))^{n-k}$  is the probability of being in the blue part. At the same time,  $\frac{n!}{(k-1)!(n-k)!}$  is a combinatorial factor since all blue and orange cells are interchangeable.

### 1.2.5. Stratification

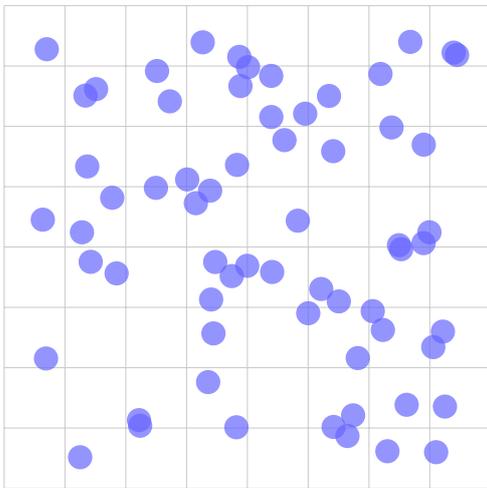


Figure 1.9.: Pure Random distribution. It shows clusters and empty spaces

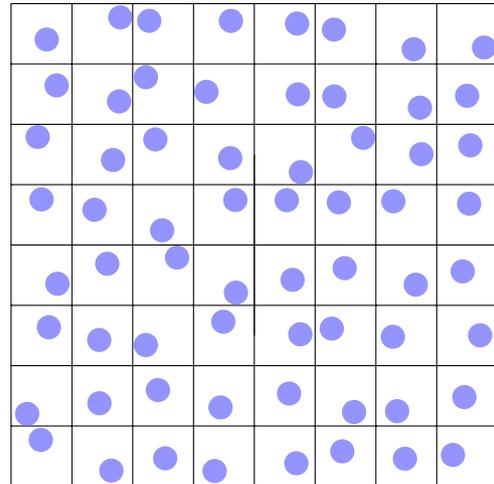


Figure 1.10.: Stratification of of the distribution. Each cell contains one particle

Stratification is a technique for achieving a lower noise level in random sampling by restricting the position of the random number to a specific interval. See the figure 1.9, which shows how the coordinates would look like if they were totally random. It shows clusters and some spots without any particles. We want, in general, to avoid something like this to gain a better distribution to get for our purpose.

The picture 1.10 shows how the distribution would look like in the case of stratified sampling. Each cell contains a photon, the photons in the cell are randomly distributed, and the distribution looks more evenly spread out. We will not go into many mathematical details, but stratification is often used and reduces the variance. For a more profound explanation, we will refer to specific books like [Rub16] or [DBB18].

### 1.3. Short Summary of Previous Work

This section will give an overview of previous work related to the theoretical study of photon mapping.

Photon mapping as algorithm goes back to 1996, in which Jensen Henrik Wann [Jen96] first published it. It was, for the time, a pretty fast algorithm. The first theoretical study of the algorithm by applying order statistics was done by García. The paper [GUS12] uses order statistics to show that the photon mapping algorithm has a slight over bias by  $\frac{k}{k-1}$ . Meaning :

$$E[\hat{I}_k] = \frac{k}{k-1}I$$

Using  $k-1$  instead of  $k$  in the algorithm does remove the over bias in the irradiance estimate. In some practical implements, this was already done, and they were using  $k-1$  or  $k-\frac{1}{2}$  photons.

The paper [GUPS14] studies the effect of kernels. The irradiance is calculated for commonly used kernels like the Constant, the Cone, the Specialized differential, the Epanechnikov, the Silverman, and the Gaussian kernel. This is an extension of the first paper to see the change in bias with kernels.

The bachelor thesis [Gan18] continues with studying kernels. The thesis looks at the variance of photon mapping with different kernels to see its effect on the variance. It has results on all the previously mentioned kernels.

Kathrin Hartmann [Har20] did study how stratification can be applied to photon mapping and how it will change the bias. She applied two different stratification strategies and calculated the irradiance estimate for those cases.

### 1.4. Comparison of Symbolical Mathematics Software

Most of the mathematical evaluations in this work were done with the help of mathematical software. Therefore, I was exploring the state of mathematical software, especially in open-source software, for symbolic evaluation and simplification of integrals. For my evaluation I was using SageMath<sup>1</sup>, SymPy<sup>2</sup>, and Mathematica<sup>3</sup> as proprietary reference. For simple integrals, all software performed fine with a few exceptions, while Mathematica is most of the cases faster than the others. For more difficult integrals, like those encountered in the theoretical part of this work, only Mathematica could solve it. From a personal perspective, Mathematica seems more user-friendly than the open-source alternatives. Offering a side panel with common functions and showing suggestions for your previous input makes it easier to use.

In conclusion, Mathematica is, in my opinion, still the best choice for this work. The open-source alternatives still need some improvements. Especial for doing manipulations on symbolic equations, the user interface of Mathematica is a great help. A great benefit of SymPy, which I later discovered, is combining it with python to do some calculations. At the

---

<sup>1</sup><https://www.sagemath.org/>

<sup>2</sup><https://www.sympy.org/>

<sup>3</sup><https://www.wolfram.com/mathematica/>

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same time, for Mathematica, you would have to learn a particular programming language to achieve the same.

The screenshot shows the Mathematica interface with several input cells and a Classroom Assistant panel on the right. The input cells contain the following code:

```

= EI[irr2d, r^2, 2 r]
=  $\frac{\int k \Gamma[-1+k]}{\Gamma[k]}$  if Re[k] < 1 + Re[n] && Re[k] > 1
= EI[irr1d, r, 1]
=  $\frac{\int k \Gamma[-1+k]}{\Gamma[k]}$  if Re[k] < 1 + Re[n] && Re[k] > 1
= EI[irr3d, r^3, 3 * r^2]
=  $\frac{\int k \Gamma[-1+k]}{\Gamma[k]}$  if Re[k] < 1 + Re[n] && Re[k] > 1
=  $\frac{\int k \Gamma[-1+k]}{\Gamma[k]}$  if Re[k] < 1 + Re[n] && Re[k] > 1
=  $\frac{\int k \Gamma[-1+k]}{\Gamma[k]}$  if Re[k] < 1 + Re[n] && Re[k] > 1
= FullSimplify[ $\frac{\int k \Gamma[-1+k]}{\Gamma[k]}$ , k ∈ Integers]
=  $\frac{\int k}{-1+k}$ 

```

The Classroom Assistant panel on the right includes a navigation menu with sections like Basic Commands, Expressions and Equations, Writing and Formatting, Typesetting, Keyboard, and Help and Settings. The Expressions and Equations section is expanded, showing options like Simplify, Solve, Expand, NSolve, Factor, FindRoot, Together, Apart, and Algebraic Trig.

Figure 1.11.: Mathematica user-interface

The screenshot shows the SageMath interface with the following input and output cells:

```

In [1]: x = var('x')
In [2]: h = sin(x)/(cos(x))^2
In [3]: h.integral(x)
Out[3]: 1/cos(x)
In [4]: h2=x^2
In [5]: h2.integral(x)
Out[5]: 1/3*x^3
In [6]: (1/(x^2+1)).integral(x)
Out[6]: arctan(x)

```

Figure 1.12.: SageMath user-interface

The screenshot shows the SymPy interface with the following code input and output cells:

```

[1]: import sympy
[2]: from sympy import *
[3]: x, y, a, b, k, n = symbols("x y a b k n")
[4]: diff(sin(x)*exp(x), x)
[5]: e^x sin(x) + e^x cos(x)
[6]: integrate(exp(x)*sin(x) + exp(x)*cos(x), x)
[7]: e^x sin(x)
[8]: integrate(sin(x**2), (x, -oo, oo))
[9]:  $\frac{\sqrt{2}\sqrt{\pi}}{2}$ 
[10]: integrate(1-x**2,(x,0,3))
[11]: -6
[12]: 1/2
[13]: 0.5
[14]: integrate(x**2/(x**4+1))
[15]:  $\frac{\sqrt{2}\log(x^2-\sqrt{2}x+1)}{8} - \frac{\sqrt{2}\log(x^2+\sqrt{2}x+1)}{8} + \frac{\sqrt{2}\operatorname{atan}(\sqrt{2}x-1)}{4} + \frac{\sqrt{2}\operatorname{atan}(\sqrt{2}x+1)}{4}$ 

```

Figure 1.13.: SymPy user-interface

## 2. Theoretical Study

In this chapter, we will deal with the theoretical framework for calculating the Variance of photon-mapping. We will start calculating some critical results in 1D; while it will not be of particular practical use, it allows for an easier understanding. We will extend it to 2D surface-illumination and 3D for light in a participating medium. In the following chapter,

<b>Photon mapping parameters</b>	
P	point where the photon mapping is done
n N	Number of photon impacts
k	Number of photon sought in Photon Maps query
$r_k$ r	Distance of the $k^{th}$ nearest photon in a query
$\Phi$	Flux
$\phi = \frac{\Phi}{n}$	Flux per Photon
<b>Distribution functions</b>	
g	Volume density of photon impacts
f	Volume density of at a given distance of the photon map query
F	Accumulated distribution of f
$F_{X(k)}$	$k^{th}$ order statistic of f
<b>Irradiance and expected values</b>	
$I$ $I(P)$	Irradiance
$\hat{I}$ $\hat{I}_k$	Photon mapping irradiance estimate
$E[X]$	Expectation value of X
$I(k) = E[\hat{I}]$	Expectation value of $\hat{I}^*$
$Var[X]$	Variance of X
<b>Quantities for Stratification</b>	
$V_a$	Volume up to point a
$V_b$	Volume up to point b
$V = V_b - V_a$	Volume difference of point a and b

Table 2.1.: Overview of Symbols. We will leave the index  $k$  away if it is clear from the context. The symbols follow the same convention as [GUPS14]. \*This notation is only used in figure

we work in spherical coordinates. For the calculations,  $r$  denotes the radius of the problem. We will not differentiate between irradiance in 1D, irradiance in 2D and power density in 3D but will commonly call them  $I$ .

## 2. Theoretical Study

### 2.1. 1D Study

To better understand how to apply the different concepts, we will look at the 1D case. This case should be understood as some simple example for demonstration purposes and not as a real-world application.

#### 2.1.1. Variance without stratification in 1D

The probability density of photon impacts is constant. The area in 1D with  $r = 1$ , which is equivalent to the interval  $[-1, 1]$ , is  $A = 2$

$$g(x) = \frac{1}{A} = 1/2 \quad (2.1)$$

Since we work in spherical coordinates we have  $g(x)dx = g(x)2dr = f(r)dr$

$$f(r) = 1 \quad (2.2)$$

The accumulated PDF<sup>1</sup> is  $F(r) = \int_0^r f(r')dr'$

$$F(r) = r \quad (2.3)$$

All the photons will carry the same flux:

$$\phi = \frac{A(r=1)I(P)}{n} = \frac{2I(P)}{n} \quad (2.4)$$

The area  $A$  is for the 1D case is given by  $A = 2r_k$ . The estimate of I is k times the flux of a single photon divided by the area of the circle up to the  $k^{th}$  Photon with radius  $r_k$ .

$$\hat{I}_k = \frac{k\phi}{A} = \frac{I k}{r n} \quad (2.5)$$

using 2.4 in 2.5 we get:

$$\hat{I}_k = \frac{I k}{r n} \quad (2.6)$$

The  $k^{th}$  order statistics distribution is given by 1.22:

$$f_k(r) = \frac{n!}{(k-1)!(n-k)!} F^{k-1}(r)(1-F(r))^{n-k} f(r) \quad (2.7)$$

Substituting in  $f_k(r)$  2.2 and  $F_k(r)$  2.3 in  $f_k(r)$

$$f_k(r) = \frac{n!}{(k-1)!(n-k)!} r^{k-1}(1-r)^{n-k} 1 \quad (2.8)$$

The expectation value of  $\hat{I}$  is

$$E[\hat{I}_k] = \int_0^1 \frac{n!}{(k-1)!(n-k)!} r^{k-1}(1-r)^{n-k} \frac{I k}{r n} dr \quad (2.9)$$

$$= \frac{I k \Gamma(k-1)}{\Gamma(k)} \quad (2.10)$$

$$= I \frac{k}{k-1} \stackrel{k \gg 1}{\approx} I + \frac{I}{k} + O\left(\frac{1}{k^3}\right) \quad (2.11)$$

---

<sup>1</sup>Probability Density Function

This shows  $E[\hat{I}]$  has a bias and is overestimate the value by  $\frac{k}{k-1}$ . It is the case in any dimensions and was proven by [GUPS14] for  $d \in (1, 2, 3)$ . For big  $k$ , we can estimate the bias as  $\frac{1}{k}$ . It shows the bias will be smaller the bigger we choose  $k$ . We could use the first-order approximation to estimate the maximum overestimation or could use it to reduce the bias. One way to reduce the bias would be to subtract  $\frac{1}{k}$  from the final result. It would leave the variance invariant because of the translation invariance (see 1.1). Alternatively, multiplying expected by  $\frac{k-1}{k}$ , we would get an unbiased result. Note that  $(I + \frac{I}{k} + O(\frac{1}{k^3})) \frac{k-1}{k} = (I + O(\frac{1}{k^3}))$  which is consistent with our approximation for big  $k$ . The multiplication  $\frac{k-1}{k}$  is equivalent to have  $k \rightarrow k-1$  in  $E_k = E[\hat{I}_{k-1}]$  while keeping the area  $\Delta A_k$  for the  $k^{\text{th}}$  photon.

The variance can now be calculated by using equation 2.4, 2.9 and 2.8

$$\text{Var}[\hat{I}] = E[\hat{I}^2] - E[\hat{I}]^2 \quad (2.12)$$

$$= \int_0^1 \frac{I^2 k^2 n! r^{k-3} (1-r)^{n-k}}{n^2 (k-1)! (n-k)!} dr - (I \frac{k}{k-1})^2 \quad (2.13)$$

$$= \frac{I^2 k^2 (-k+n+1)}{(k-2)(k-1)^2 n} \quad (2.14)$$

$$= -\frac{I^2 k^3}{(k-2)(k-1)^2 n} + \frac{I^2 k^2}{(k-2)(k-1)^2 n} + \frac{I^2 k^2}{(k-2)(k-1)^2} \quad (2.15)$$

$$\stackrel{n \gg 1, k \gg 1}{\sim} \frac{I^2}{k} - \frac{I^2}{n} + O(\frac{1}{nk}) \sim \frac{I^2}{k} \quad (2.16)$$

It shows that the variance goes with  $I^2/k$  to zero. We are assuming  $n \gg k$  in the last step. The step before shows that if  $k$  gets close to  $n$ , the variance reaches even faster to zero.

### 2.1.2. Variance with stratification in 1D

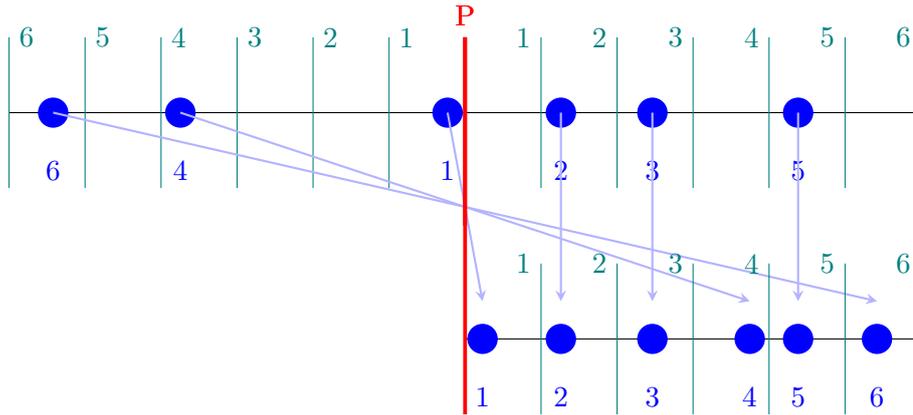


Figure 2.1.: Stratification: Ordering the stratified sample by distance from the origin. In the above example, the photons are not ordered by distance. The below pictures illustrates the mapping to distance

## 2. Theoretical Study

We now repeat the calculation with stratification. Each photon should be in its own shell<sup>2</sup> and each shell goes from  $[\frac{i-1}{n}, \frac{i}{n}]$  with  $n$  being the maximal number of photons. The position of each photon is mapped to the distance from the point P. See figure 2.1 for an illustration. The order statistics is elementary for this case. For  $n = 1$  and  $k = 1$  the formula 1.22 simplifies to:

$$f(r) = \frac{1}{\frac{k}{n} - \frac{k-1}{n}} = n$$

We can directly calculate the expectation value

$$E[\hat{I}] = \int_{\frac{k-1}{n}}^{\frac{k}{n}} \frac{I k}{r n} n dr \quad (2.17)$$

$$= I k (Ln(k) - Ln(k-1)) \quad (2.18)$$

$$= I k Ln\left[\frac{k}{k-1}\right] \quad (2.19)$$

$$= I + \frac{I}{2k} + \frac{I}{3k^2} + O\left(\left(\frac{1}{k}\right)^3\right) \quad (2.20)$$

This intermediate result is in accordance with the finding of Hartmann [Har20], which showed that for this particular stratification strategy, the expectation value goes with  $ln$ . Making the exact approximation for large  $k$ , we see that the bias only goes with  $\frac{1}{2k}$ . This is a novel theoretical justification of the comment in [GUPS14] ‘‘Some implementations have used  $k-0.5$  when all photons carry the same flux (Dr. Per Christensen, private communication, 2012).’’ The additional factor would be equivalent to subtract only half the quantity we did in the case without stratification.

The variance can be calculated in the same way.

$$Var[\hat{I}] = E[\hat{I}^2] - E[\hat{I}]^2 \quad (2.21)$$

$$= \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left(\frac{I k}{r n}\right)^2 n dr - (I k (-Ln[k/n] + Ln[(1+k)/n]))^2 \quad (2.22)$$

$$= (I^2 k^2)/(k+k^2) - I^2 k^2 (-Ln[k/n] + Ln[(1+k)/n])^2 \quad (2.23)$$

$$= \frac{I^2}{12k^2} + O\left(\left(\frac{1}{k}\right)^3\right) \quad (2.24)$$

It should be pointed out that the variance goes with  $1/k^2$  to zero, while without stratification, it only did go with  $1/k$ . The term  $\frac{I^2}{12k^2}$  can be used to estimate the variance.  $Var \leq C(k^*) \frac{I^2}{12k^2}$  for all  $k > k^*$ .  $k^*$  is a fixed but arbitrary number given the lower bound for which this inequality holds.

### 2.1.3. Variance for multiple photons in one strata 1D

We now take a look at a more general case by allowing more photons per shell. Figure 2.2 shows how it could look like.

<sup>2</sup>Shell  $p$  is in this context the area of the interval in which the photon  $I$  can be

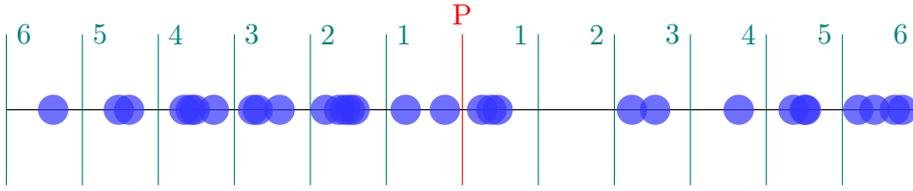


Figure 2.2.: Stratification Example

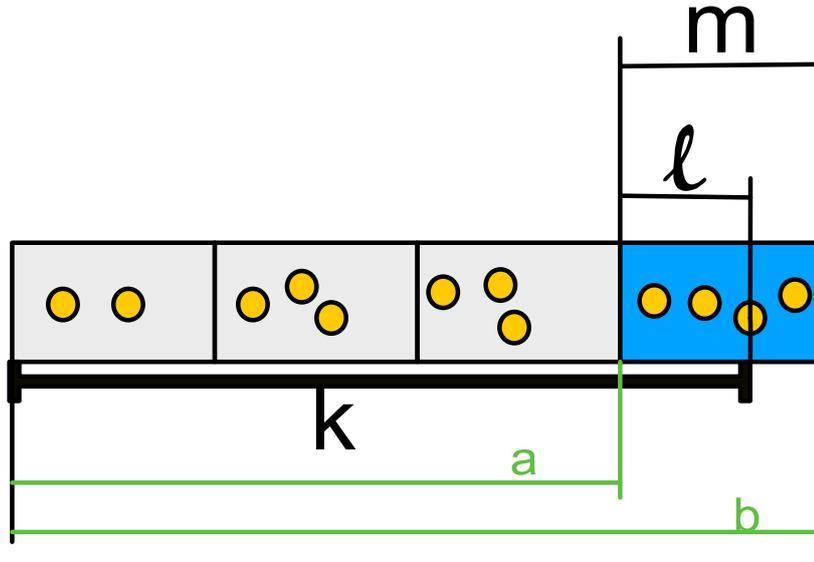


Figure 2.3.:  $l$  counts the number of Photon beginning in the strata in which the  $k$ th Photon is  
 $m$  is the total number of photons of the strata in which the photon  $k$  is  
 $V_a$  is the whole gray area  
 $V = V_b - V_a$  is the blue area  
 $V_b$  is the whole gray area+blue area, e.g., all the area up to including the last strata which include  $k$

For making the calculation easier, we introduce the following variables (see 2.3):

- $V_a$  Volume up to a point  $a$
- $V_b$  Volume up to the point  $b$  while point  $b$  is further away than point  $a$
- $V$  as the difference between the two Volumes

For the most general case, we begin by using the equation

$$E(\hat{I}) = \int_0^1 dr \theta(a < r < b) f_k(r) \hat{I}_k = \int_a^b dr f_k(r) \hat{I}_k \quad (2.25)$$

Here the  $\theta(x)$  function is 1 if the arguments evaluates to true and 0 otherwise. The  $k^{th}$  order

## 2. Theoretical Study

statistic distribution is given by 1.22:

$$f_k(r) = \frac{m!}{(l-1)!(m-l)!} F^{k-1}(r)(1-F(r))^{n-k} f(r) \quad (2.26)$$

$$= CF^{k-1}(r)(1-F(r))^{n-k} f(r) \quad (2.27)$$

Notice that we are now ordering the photons inside one shell and thus are using not  $k$  and  $n$  but  $l$  and  $m$  for the order statistics parameters. We have  $\hat{I}_k = \frac{k\phi}{A} = \frac{I}{r} \frac{k}{n}$  from 2.69 and  $F(r) = \frac{r-a}{b-a}$

$$\begin{aligned} E(\hat{I}) &= \int_a^b dr \frac{Ik m! \left(\frac{r-a}{b-a}\right)^{l-1} \left(1 - \frac{r-a}{b-a}\right)^{m-l}}{nr(l-1)!(b-a)(m-l)!} \\ &= IkC \int_0^1 \frac{x^{l-1}(1-x)^{m-l}}{Vx+a} dx \\ &= \frac{Ik {}_2F_1\left(1, l; m+1; -\frac{V}{a}\right)}{V_a} \end{aligned} \quad (2.28)$$

For how to transform the integral in this form, see the appendix A.3.2. We use the integral representation of the hypergeometric function to solve the integral in terms of special functions.

$$E(\hat{I}) = \frac{Ik {}_2F_1\left(1, l; m+1; -\frac{V}{a}\right)}{V_a} \quad (2.29)$$

For the variance, the expectation value of  $I^2$  is needed too. By using the same integral transformation and variables we were using for the integral of  $E(\hat{I})$ .

$$E(I^2) = \frac{I^2 k^2 \left( (V_a(-m) + V_a - lV + V) {}_2F_1\left(1, l; m+1; -\frac{V}{a}\right) + V_a m \right)}{V_a^2 (V_a + V)} \quad (2.30)$$

The end result of the variance is:

$$\begin{aligned} Var(I) &= E(I^2) - E(I)^2 \\ &= \frac{I^2 k^2 \left( (V_a(-m) + V_a - lV + V) {}_2F_1\left(1, l; m+1; -\frac{V}{a}\right) + V_a m \right)}{V_a^2 (V_a + V)} \\ &\quad - \frac{I^2 k^2 {}_2F_1\left(1, l; m+1; -\frac{V}{a}\right)^2}{V_a^2} \end{aligned} \quad (2.31)$$

This result is quite general as we later will show that this formula will hold in any dimensions and not just in 1D. We will not further evaluate the formula but will make better use of it in the more realistic cases in the next section.

## 2.2. 2D Study

### 2.2.1. Variance without Stratification in 2D

After doing the calculations in the 1D case, we can now apply the same principle to the more practical 2D case (surface illumination). The photon distribution is again a constant function and given by:

$$g(x) = \frac{1}{\pi} \quad (2.32)$$

The factor  $\pi$  comes from the fact that the unit disc of radius one has the area  $\pi$ , in contrast to the 1D case with  $\Delta A = 2$ . For the 2D case, we have for the probability density function:

$$f(x) = 2r \quad (2.33)$$

and

$$F(x) = \int_0^r f(r)dx = r^2 \quad (2.34)$$

We are working in spherical coordinates. Therefore, we need to keep in mind that with  $r$ , the surface changes with  $r$ , and the area changes with  $r^2$ . This is reflected in the formula for  $F$  and  $f$ . Strictly speaking, it transforms probability density. The irradiance estimate is given again by  $\frac{I \text{Area}(r=1) k}{\Delta \text{Area} n}$

$$\hat{I} = \frac{I k}{r^2 n} \quad (2.35)$$

To ensure that we are dealing with a valid probability density, the functions need to be normalized.

$$1 = \int_{S^2} f(r)dr \quad (2.36)$$

Plugging in the previous equations in the formula 1.22

$$f_k(r) = \frac{n!}{(k-1)!(n-k)!} (r^2)^{k-1} (1-r^2)^{n-k} 2r \quad (2.37)$$

As a reminder, the expectation value is given by

$$E[\hat{I}] = \int_0^1 dr_k \hat{I} f_k(r_k) \quad (2.38)$$

the expectation value can be calculated:

$$E[\hat{I}_k] = \int_0^1 \frac{2Ik n!}{n(k-1)!(n-k)!} \frac{(r^2)^{k-1} (1-r^2)^{n-k}}{r} dr \quad (2.39)$$

$$= \frac{Ik \Gamma(k-1)}{\Gamma(k)} = I \frac{k}{k-1} \stackrel{k \gg 1}{\approx} I + \frac{I}{k} \quad (2.40)$$

## 2. Theoretical Study

For the variance, we can use the value for the expectation value and calculate the same integral with just  $\hat{I}^2$

$$\text{Var}[\hat{I}_k] = E[\hat{I}^2] - E[\hat{I}]^2 \quad (2.41)$$

$$= \frac{I^2 k^2 (-k + n + 1)}{(k - 2)(k - 1)^2 n} \quad (2.42)$$

$$\underset{n \gg k \gg 1}{\sim} \frac{I^2}{k} \quad (2.43)$$

We can see, we get the same result for the 2D case as in the 1D case. The result agrees with the result in [Gan18]. The expectation value is overestimating the ground truth by around  $\frac{I}{k}$ . This could be interpreted as overestimating the ground truth by one photon, carrying  $\frac{I}{k}$  irradiance.

### 2.2.2. Variance with Stratification in 2D

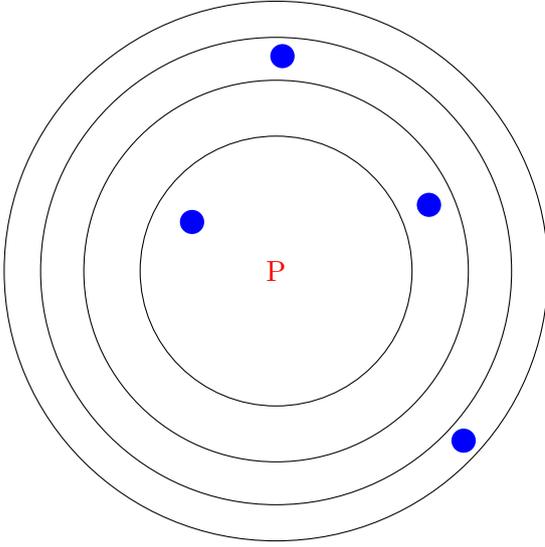


Figure 2.4.: Stratification 2D

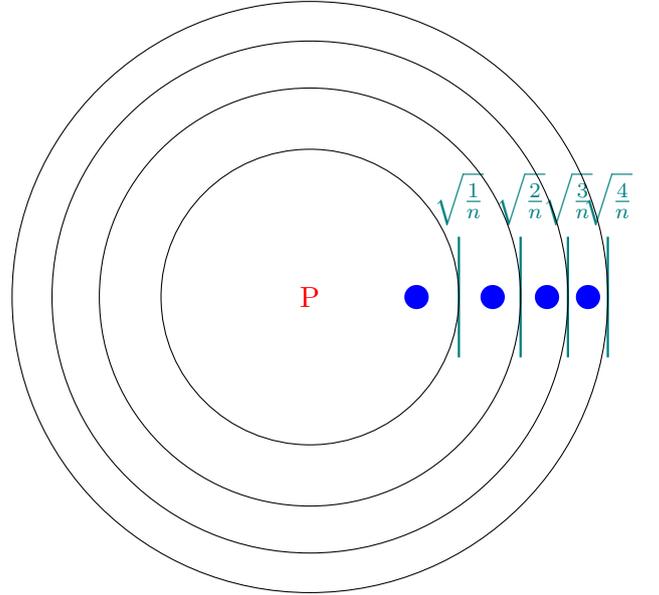


Figure 2.5.: Ordered by distance

We put one photon per strata, and to keep the photon-density constant, we need to adjust each radius to keep the volume per strata fixed. We fix the volume of shells to  $\pi$  in this example. We do this by choosing the radius of the shells to be  $r = \sqrt{i/n}$ . The figure 2.6 does illustrate this quantization.

$$f_k(x) = \frac{m!}{(l-1)!(m-l)!} F^{l-1}(x) (1-F(x))^{m-l} f(x)$$

does again simply to  $f(x)$  for the case of only having one photon per strata. e.g.  $l = 1$  and  $m = 1$ . Like in the previous section  $F(x) = r^2$ ,  $f(x) = 2r$  and  $\hat{I} = \frac{I}{r^2} \frac{k}{n}$

$$E[\hat{I}_k] = \int_0^1 \hat{I} f_k(r) \theta\left(\sqrt{\frac{k-1}{n}} < r < \sqrt{\frac{k}{n}}\right) dr \quad (2.44)$$

$$= \int_{\sqrt{\frac{k-1}{n}}}^{\sqrt{\frac{k}{n}}} \frac{2Ik}{r} dr \quad (2.45)$$

$$= I k (Ln(k) - Ln(k-1)) \quad (2.46)$$

$$= I k Ln\left(\frac{k}{k-1}\right) \quad (2.47)$$

$$= I + \frac{I}{2k} + \frac{I}{3k^2} + O\left(\left(\frac{1}{k}\right)^3\right) \quad (2.48)$$

The result agrees with the work of Hartmann [Har20]. As expected, we are getting the same result as in the 1D case with a bias of  $\sim \frac{1}{2k}$ . We can see that in this stratification strategy, simply subtracting  $\frac{1}{2}$  photon carrying  $I/k$  irradiance would reduce the bias. This could be seen as some theoretical explanation on why it could work well in practice to only calculate the expectation value with  $k - \frac{1}{2}$  instead of  $k$  as mentioned in [GUPS14]. The variance can be evaluated the same way.

$$Var[\hat{I}_k] = I^2 k \left( \frac{1}{k-1} - k Ln^2\left(\frac{k-1}{k}\right) \right) \quad (2.49)$$

$$= \frac{I^2}{12k^2} + O\left(\left(\frac{1}{k}\right)^3\right) \quad (2.50)$$

Again, expectation and variance is independence of the dimension on which the calculation was carried out.

### 2.2.3. General Case

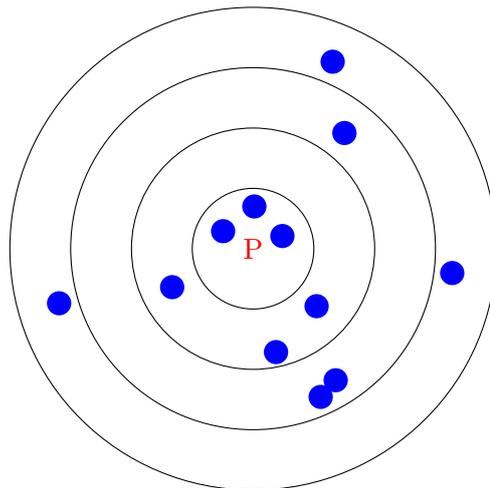


Figure 2.6.: Example distribution of multiple photons per shell in 2D

## 2. Theoretical Study

We will solve the most general case in 2D. The variables we will use here are the same as in 1D. 2.1.3 For this dimension the only difference is the appropriate volume (Area).

$$V_a = a^2 \quad (2.51)$$

$$V_b = b^2 \quad (2.52)$$

$$V = b^2 - a^2 \quad (2.53)$$

For an explanation of the variables see 2.3 We start with the most basic equation:

$$E(\hat{I}) = \int_0^1 dr \theta(a < r < b) f_k(r) \hat{I}_k = \int_a^b dr f_k(r) \hat{I}_k \quad (2.54)$$

with the use of the  $k^{th}$  order statistic distribution 1.22:

$$f_k(r) = \frac{m!}{(l-1)!(m-l)!} F^{k-1}(r) (1-F(r))^{n-k} f(r) \quad (2.55)$$

$$= C F^{k-1}(r) (1-F(r))^{n-k} f(r) \quad (2.56)$$

we have  $\hat{I}_k = \frac{k\phi}{A} = \frac{l}{r^2} \frac{k}{n}$  from 2.69 and  $F(r) = \frac{r^2-a^2}{b^2-a^2}$

$$\begin{aligned} E(\hat{I}) &= \int_a^b dr \frac{Ik m! \left(\frac{r^2-a^2}{b^2-a^2}\right)^{l-1} \left(1 - \frac{r^2-a^2}{b^2-a^2}\right)^{m-l}}{nr(l-1)!(b-a)(m-l)!} \\ &= Ik \int_0^1 \frac{x^{l-1} (1-x)^{m-l}}{Vx + V_a} dx \\ &= \frac{Ik {}_2F_1\left(1, l; m+1; -\frac{V}{a^2}\right)}{a^2} \end{aligned} \quad (2.57)$$

As we can see, we can bring the formula into the same form as in the 1D case. For more details, one can again see the appendix A.3.1.

$$E(\hat{I}) = \frac{Ik {}_2F_1\left(1, l; m+1; -\frac{V}{V_a}\right)}{V_a} \quad (2.58)$$

The  $E(\hat{I}^2)$  evaluates in the same way.

$$E(\hat{I}^2) = \frac{I^2 k^2 \left( (V_a(-m) + V_a - lv + v) {}_2F_1\left(1, l; m+1; -\frac{v}{V_a}\right) + V_a m \right)}{V_a^2 (V_a + v)} \quad (2.59)$$

The final result of the variance is:

$$\begin{aligned} Var(\hat{I}) &= E(\hat{I}^2) - E(\hat{I})^2 \\ &= \frac{I^2 k^2 \left( (V_a(-m) + V_a - lV + V) {}_2F_1\left(1, l; m+1; -\frac{V}{V_a}\right) + V_a m \right)}{V_a^2 (V_a + v)} \\ &\quad - \frac{I^2 k^2 {}_2F_1\left(1, l; m+1; -\frac{V}{V_a}\right)^2}{V_a^2} \end{aligned} \quad (2.60)$$

These formulas are quite general and can be applied to arbitrary stratification strategies.

### 2.2.4. Example Stratification

In this section, we will use this formula to evaluate some stratification strategies.

#### Stratification Strategy 1

We can reduce our general formula to the special case in which only one photon per shell. We will make use of the following relation:

$${}_2\tilde{F}_1(1, 1; 1 + 1; V/V_a) = \frac{V_a \text{Ln}(V_b/V_a)}{V} \quad (2.61)$$

For this case, we have  $l = m = 1$ , and by using this in our formula for the expectation value, we are getting:

$$\lim_{m,l \rightarrow 1} E(\hat{I}) = Ik \text{Ln}\left(\frac{k}{k-1}\right) \quad (2.62)$$

The simplifications were carried out with the help of Mathematica and see the Appendix B

#### Stratification Strategy 2

Another stratification strategy is splitting the area into equal-sized regions and putting one photon in one region each. The reason behind this: For bigger  $k$ , the shell becomes very thin, and this way, we are getting bigger shells and split the area into equal-sized regions to keep the photon density the same. The principle is illustrated in the following figure 2.7 and 2.8. The  $p^{th}$  shell contains  $p$  photons, has a area of  $p \pi$  and an outer/inner radius  $r_{out}^2 = \frac{(p+1)(p+2)}{2} / r_{inner}^2 = \frac{(p)(p+1)}{2}$ . We can express  $V_a$  and  $V_b$  in terms of  $k - k^{th}$  closest photon which is in shell  $p$ ,  $l - k^{th}$  photon is the  $l^{th}$  photon in the shell  $p - anm$  - total amount of photons in the  $p^{th}$  shell. We can now calculate the expected value and variance in a simple computer program.

## 2. Theoretical Study

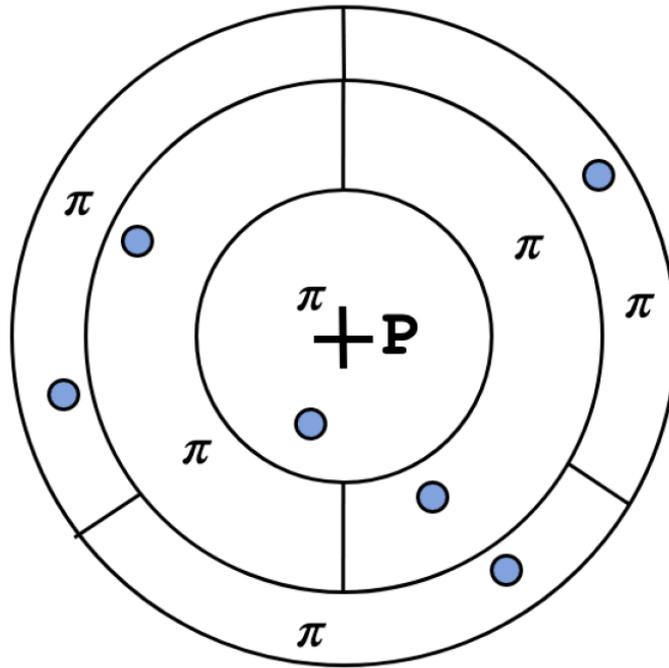


Figure 2.7.: Stratification in 2D with more photons per shell but fixed area per photon - Image by courtesy of Kathrin Hartmann [Har20]

”Calculation of the expected value and variance”

```
1      m_max
2
3      # empty list
4      # for expectation values
5      lst=[]
6      # for variance
7      vlst=[]
8
9      k=1
10     for m in range(1,m_max):
11         for l in range(1,m+1):
12             k+=1
13
14             # the difference in volume between point a and b is m
15             v=m
16             # volume of point a is just k-l
17             a=k-l
18             # expectation value
19             e1=N(hyper((1, l), [1+m], -v/(a)))*k/(a)
20             # variance
21             v1=N(var(l,m,k))
22             # append to list for different k values
23             vlst+=[v1]
24             lst+=[e1]
```

Figure 2.13 shows the result of this calculation. We can see that the expected value divided

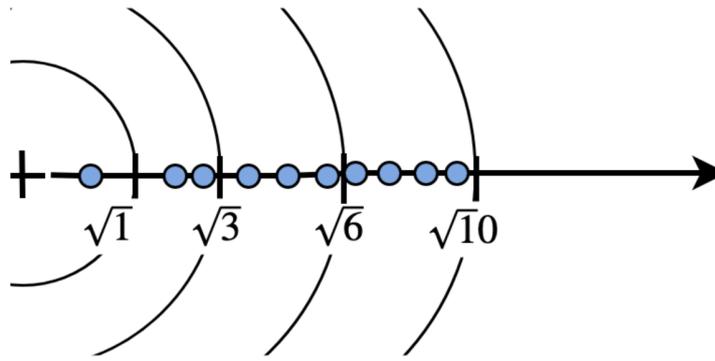


Figure 2.8.: Stratification in 2D with fixed area per photon projected to a line. Note that we impose other scaling to the problem as Kathrin.  $R_{max} = 1$  in contrast to  $R_{max} = \sqrt{N}$  used by Kathrin -Image by courtesy of Kathrin Hartmann

by the real value goes to 1 fast for  $k$  big enough. We can see the same pattern in the expectation value as already observed in [Har20], and for the variance, we get a similar oscillating pattern. The expectation value seems to always have a local minimum at the first photon at a new shell and the local maximum at the last photon at the same shell. It would be interesting to investigate the reason behind this further.

## 2.3. 3D Study

We will study the 3D case in this section, which describes light in a participating medium.

### 2.3.1. Variance without stratification in 3D

We begin again with calculation the uniform distribution of the photons.

$$g(x) = \frac{1}{\frac{3}{4}\pi} \quad (2.63)$$

The above equation is again in cartesian coordinates, and we need to transform it to spherical coordinates. The probability density function is given by:

$$f(r) = 3r^2 \quad (2.64)$$

with the corresponding accumulated probability density function:

$$F(x) = \int_0^r f(r)dx = r^3 \quad (2.65)$$

The probability density function is normalized by the integral over the whole sphere:

$$1 = \int_{S^3} f(x)d^3x \quad (2.66)$$

## 2. Theoretical Study

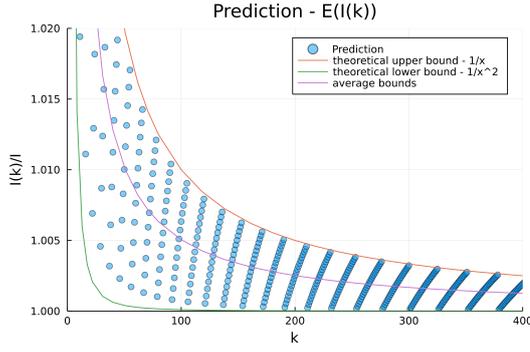


Figure 2.9.: Expectation Value of  $\hat{I}$

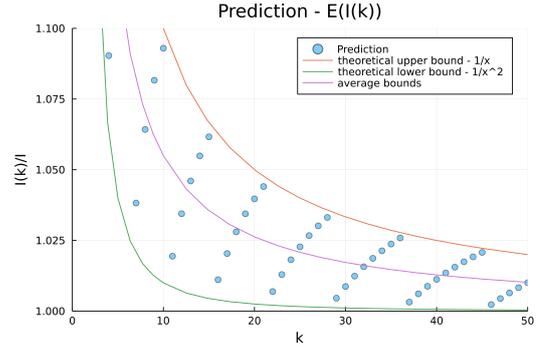


Figure 2.10.: Expectation value for small k

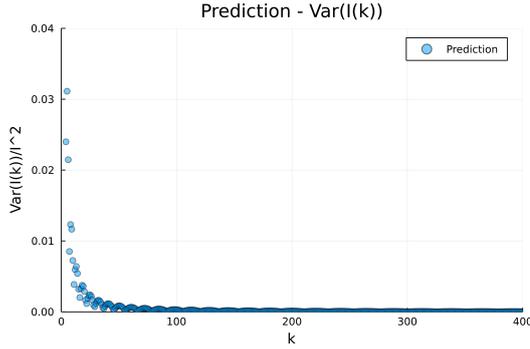


Figure 2.11.: Variance

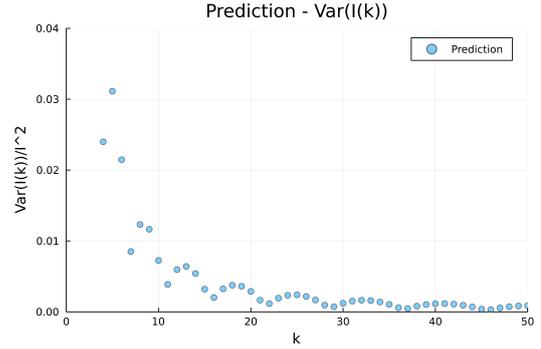


Figure 2.12.: Variance value for small k

Figure 2.13.: Expectation value and variance for stratification strategy 2

For the estimator of the power density, we use the formula:

$$\hat{I} = \frac{I k}{r^3 n} \quad (2.67)$$

in which we used:

$$\phi = \frac{A(r=1)I(P)}{n} \quad (2.68)$$

and

$$\hat{I}_k = \frac{k\phi}{A} = \frac{I k}{r^3 n} \quad (2.69)$$

Applying the order statistics formula with our probabilities functions  $f(x)$  and  $F(x)$ , we get:

$$\begin{aligned} f_k(r) &= \frac{n!}{(k-1)!(n-k)!} F^{k-1}(r)(1-F(r))^{n-k} f(r) \\ &= \frac{n!}{(k-1)!(n-k)!} (r^3)^{k-1} (1-r^3)^{n-k} 3r^2 \end{aligned} \quad (2.70)$$

For the expected value, we get the following result:

$$E[\hat{I}_k] = \frac{Ik\Gamma(k-1)}{\Gamma(k)} = I \frac{k}{k-1} \stackrel{k \gg 1}{\approx} I + \frac{I}{k} \quad (2.71)$$

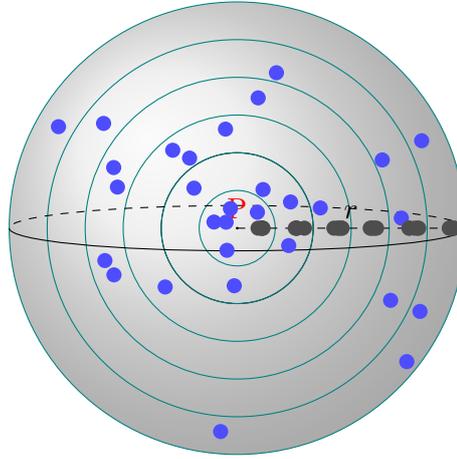
and for the variance we get:

$$Var[\hat{I}_k] = E[\hat{I}_k^2] - E[\hat{I}_k]^2 = \frac{I^2 k^2 (-k + n + 1)}{(k - 2)(k - 1)^2 n} \underset{n \gg 1}{\approx} \frac{I^2}{k} \quad (2.72)$$

As already expected from the previous examples, the predicted value slightly overestimates the actual value by  $\sim 1/k$ . The same goes for variance, which goes with  $\sim 1/k$  to zero. It shows again that all calculations are independent of the dimensions.

### 2.3.2. Variance with stratification in 3D

We will allow for only one photon per shell and reduce the inner and outer radius of the shells to ensure a uniform distribution. This concept is illustrated in the following figures 2.14 and 2.15. The outer radius of shell  $p$  is  $r = \sqrt[3]{\frac{p}{n}}$ . The order statistics does simply for the



case of only one photon per strata.

$$f_k(x) = \frac{m!}{(l-1)!(m-l)!} F^{l-1}(x) (1-F(x))^{m-l} f(x) = f(x) \quad (2.73)$$

The expectation value can now calculate straightforwardly:

$$E[\hat{I}_k] = \int_0^1 \hat{I} f_k(r) \theta \left( \sqrt[3]{\frac{k-1}{n}} < r < \sqrt[3]{\frac{k}{n}} \right) dr \quad (2.74)$$

$$= \int_{\sqrt[3]{\frac{k-1}{n}}}^{\sqrt[3]{\frac{k}{n}}} \frac{3Ik}{r} dr \quad (2.75)$$

$$= I k (Ln(k) - Ln(k-1)) \quad (2.76)$$

$$= I + \frac{I}{2k} + \frac{I}{3k^2} + O\left(\left(\frac{1}{k}\right)^3\right) \quad (2.77)$$

## 2. Theoretical Study

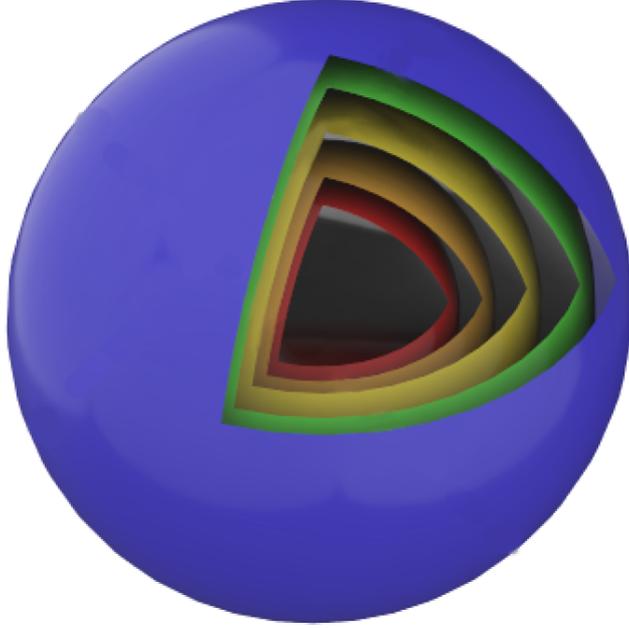


Figure 2.14.: Example of stratification in 3D with one photon per strata - Image by courtesy of Kathrin Hartmann [Har20]

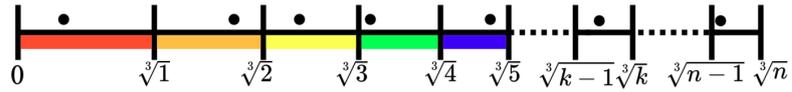


Figure 2.15.: Stratification 3D - 1 photon per strata projected to a line - Image by courtesy of Kathrin Hartmann

The same goes for the variance.

$$Var[\hat{I}_k] = \frac{I^2 k}{k-1} - I^2 k^2 \left( Ln \left( \frac{k}{n} \right) - Ln \left( \frac{k-1}{n} \right) \right)^2 \quad (2.78)$$

$$= \frac{I^2}{12k^2} + O \left( \left( \frac{1}{k} \right)^3 \right) \quad (2.79)$$

The final result shows again that the variance goes faster  $\sim 1/k^2$  to 0 as the in the non-stratification case  $\sim 1/k$ . The expected value overestimates the true value by  $\sim 1/k$ .

### 2.3.3. General Case

The most general case with  $m$  photons in a shell works the same way as in the 2D and 1D cases. We need to adjust our definitions of the volumes to reflect the change to 3D. For the

volumes, we have:

$$V_a = a^3 \quad (2.80)$$

$$V_b = b^3 \quad (2.81)$$

$$V = b^3 - a^3 \quad (2.82)$$

The expectation value of the estimator:

$$E(\hat{I}) = \int_0^1 dr \theta(a < r < b) f_k(r) \hat{I}_k = \int_a^b dr f_k(r) \hat{I}_k \quad (2.83)$$

The probability of finding the  $k^{th}$  photon outside of the interval  $[a, b]$  is zero. The  $k^{th}$  order statistic distribution is given by 1.22:

$$f_k(r) = \frac{m!}{(l-1)!(m-l)!} F^{k-1}(r) (1-F(r))^{n-k} f(r) \quad (2.84)$$

$$= C F^{k-1}(r) (1-F(r))^{n-k} f(r) \quad (2.85)$$

we have  $\hat{I}_k = \frac{k\phi}{A} = \frac{l}{r} \frac{k}{n}$  from 2.69 and  $F(r) = \frac{r^3 - a^3}{b^3 - a^3}$

$$\begin{aligned} E(\hat{I}) &= \int_a^b dr \frac{Ik m! \left(\frac{r^3 - a^3}{b^3 - a^3}\right)^{l-1} \left(1 - \frac{r^3 - a^3}{b^3 - a^3}\right)^{m-l}}{nr(l-1)!(b^3 - a^3)(m-l)!} \\ &= IkC \int_0^1 \frac{x^{l-1} (1-x)^{m-l}}{vx + V_a} dx \\ &= \frac{Ik {}_2F_1\left(1, l; m+1; -\frac{V}{V_a}\right)}{V_a} \end{aligned} \quad (2.86)$$

For how to transform the integral in this form, see the appendix. We use the integral representation of the hypergeometric function to solve the integral in terms of special functions.

$$E(\hat{I}) = \frac{Ik {}_2F_1\left(1, l; m+1; -\frac{V}{V_a}\right)}{V_a} \quad (2.87)$$

For the variance, the expectation value of  $I^2$  is needed too. Using the same integral transformation and variables:

$$E(\hat{I}^2) = \frac{I^2 k^2 \left( (V_a(-m) + V_a - lV + V) {}_2F_1\left(1, l; m+1; -\frac{V}{V_a}\right) + V_a m \right)}{V_a^2 (V_a + V)} \quad (2.88)$$

The end result of the variance is:

$$\begin{aligned} Var(\hat{I}) &= E(I^2) - E(I)^2 \\ &= \frac{I^2 k^2 \left( (V_a(-m) + V_a - lV + V) {}_2F_1\left(1, l; m+1; -\frac{V}{V_a}\right) + V_a m \right)}{V_a^2 (V_a + V)} \\ &\quad - \frac{I^2 k^2 {}_2F_1\left(1, l; m+1; -\frac{V}{V_a}\right)^2}{V_a^2} \end{aligned} \quad (2.89)$$

This concludes the prove that our calculations are independent of the dimensions (1D, 2D or 3D).

## 2. Theoretical Study

### 2.3.4. Example Stratification

#### Stratification Strategy 1

In the 3D case, we can use the same simplification we already used in the previous discussion and arrive again at the same result.

$$\lim_{m,l \rightarrow 1} E(\hat{I}) = IkLn\left(\frac{k}{k-1}\right) \quad (2.90)$$

#### Stratification Strategy 2

The second strategy is shown in figure 2.16 and 2.17. The idea is to give each photon a volume of the same size in the sphere. Since the calculation of the general formula is independent of the dimensions, we will get here the same result as in the 2D case. To avoid repeating, we refer the reader to the 2D case. 2.2.4

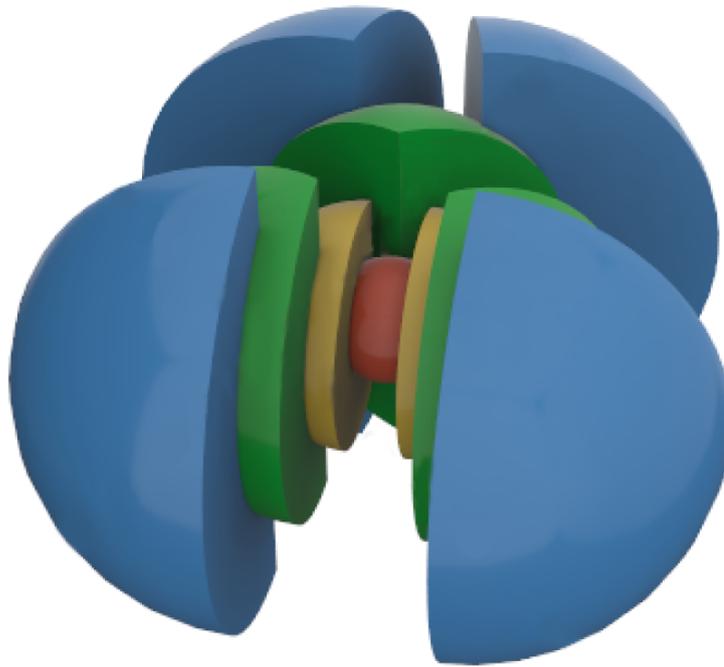


Figure 2.16.: Stratification 3D - Example of multiple photons per shell - Image by courtesy of Kathrin Hartmann

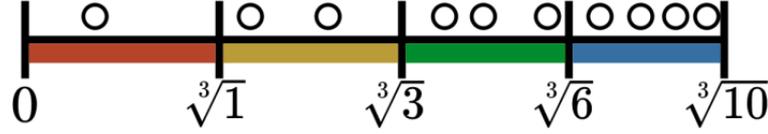


Figure 2.17.: Stratification 3D - Example of multiple photons per shell - Image by courtesy of Kathrin Hartmann

## 2.4. Summary

This section will summarize the results of the previous theoretical analysis.

Dimensions	1D	2D	3D
$\Delta A$	$2r$	$\pi r^2$	$\frac{4}{3}\pi r^3$
$\Phi$	$\frac{2I(P)}{n}$	$\frac{\pi I(P)}{n}$	$\frac{\frac{4}{3}\pi I(P)}{n}$
$\hat{I}$	$\frac{I(P)}{r} \frac{k}{n}$	$\frac{I(P)}{r^2} \frac{k}{n}$	$\frac{I(P)}{r^3} \frac{k}{n}$

Table 2.2.: Overview of quantities in different dimensions

### 2.4.1. Order statistic

Applying the order statistics to the algorithm gives the following results:

$$E[\hat{I}_k] = I \frac{k}{k-1} \stackrel{k \gg 1}{\approx} I + \frac{I}{k} \quad (2.91)$$

$$Var[\hat{I}_k] = \frac{I^2 k^2 (-k + n + 1)}{(k-2)(k-1)^2 n} \quad (2.92)$$

$$\stackrel{k \gg 1}{\approx} \frac{I^2}{k} \quad (2.93)$$

### 2.4.2. Stratification General

Applying stratification with  $m$  photon in one shell and  $k$  is the  $l^{th}$  photon in this shell: Expectation value:

$$E(\hat{I}_k) = \frac{Ik {}_2F_1\left(1, l; m+1; -\frac{V}{V_a}\right)}{V_a} \quad (2.94)$$

## 2. Theoretical Study

Variance:

$$\begin{aligned} \text{Var}(\hat{I}_k) &= E(\hat{I}_k^2) - E(\hat{I}_k)^2 \\ &= \frac{I^2 k^2 \left( (V_a(-m) + V_a - lV + V) {}_2F_1 \left( 1, l; m+1; -\frac{V}{V_a} \right) + V_a m \right)}{V_a^2 (V_a + V)} \\ &\quad - \frac{I^2 k^2 {}_2F_1 \left( 1, l; m+1; -\frac{V}{V_a} \right)^2}{V_a^2} \end{aligned} \quad (2.95)$$

### 2.4.3. Stratification Strategy 1

Allowing one photon per shell: Expectation value:

$$E[\hat{I}_k] = I k L n \left( \frac{k}{k-1} \right) \quad (2.96)$$

$$= I + \frac{I}{2k} + \frac{I}{3k^2} + O \left( \left( \frac{1}{k} \right)^3 \right) \quad (2.97)$$

Variance:

$$\text{Var}[\hat{I}_k] = I^2 k \left( \frac{1}{k-1} - k L n^2 \left( \frac{k-1}{k} \right) \right) \quad (2.98)$$

$$= \frac{I^2}{12k^2} + O \left( \left( \frac{1}{k} \right)^3 \right) \quad (2.99)$$

### 2.4.4. Stratification Strategy 2

Keeping the area fixed per photon to  $\pi$  and filling the shells up with in the – first: one photon, second: 2 photons, third: 3 photons...

The volume difference is just m:

$$V = m \quad (2.100)$$

Volume at point a is:

$$V_a = k - l \quad (2.101)$$

There is a general formula to calculate the l and m corresponding to a specific k. Still, it is easy to find the variables even without a concrete formula by just using a simple program.

The Expectation value:

$$E(\hat{I}_k) = \frac{I k {}_2F_1 \left( 1, l; m+1; -\frac{V}{V_a} \right)}{V_a} \quad (2.102)$$

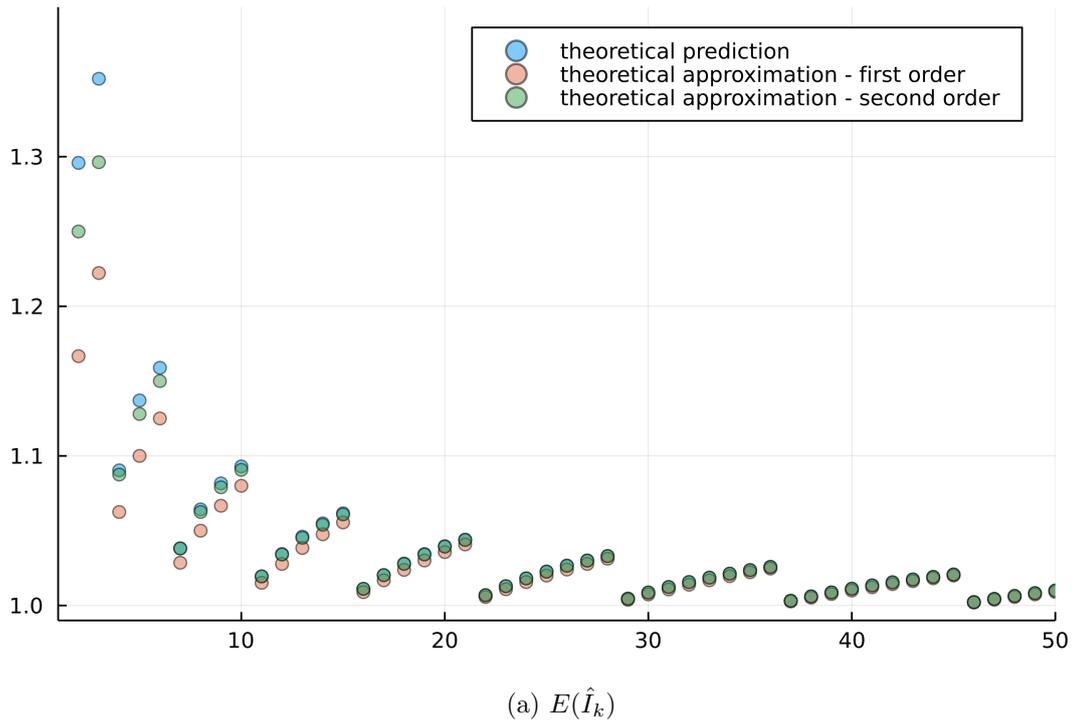
$$E(\hat{I}_k) = I + \frac{I \left( l - \frac{lv}{m+1} \right)}{k} + \frac{I \left( -\frac{(2l^2)v}{m+1} + l^2 + \frac{(l(l+1))v^2}{(m+1)(m+2)} \right)}{k^2} + O \left( \left( \frac{1}{k} \right)^3 \right) \quad (2.103)$$

The Variance:

$$\begin{aligned}
 \text{Var}(\hat{I}_k) &= E(\hat{I}_k^2) - E(\hat{I}_k)^2 \\
 &= \frac{I^2 k^2 \left( (V_a(-m) + V_a - lV + V) {}_2F_1 \left( 1, l; m+1; -\frac{V}{V_a} \right) + V_a m \right)}{V_a^2 (V_a + V)} \\
 &\quad - \frac{I^2 k^2 {}_2F_1 \left( 1, l; m+1; -\frac{V}{V_a} \right)^2}{V_a^2}
 \end{aligned} \tag{2.104}$$

$$\text{Var}(\hat{I}_k) = \frac{I^2 l v^2 (-l + m + 1)}{k^2 (m + 1)^2 (m + 2)} - \frac{4 (I^2 l v^2 (l - m - 1) (l (m - v + 3) - v))}{k^3 ((m + 1)^2 (m + 2) (m + 3))} + O \left( \left( \frac{1}{k} \right)^4 \right) \tag{2.105}$$

### Irradiance Estimator



### Variance

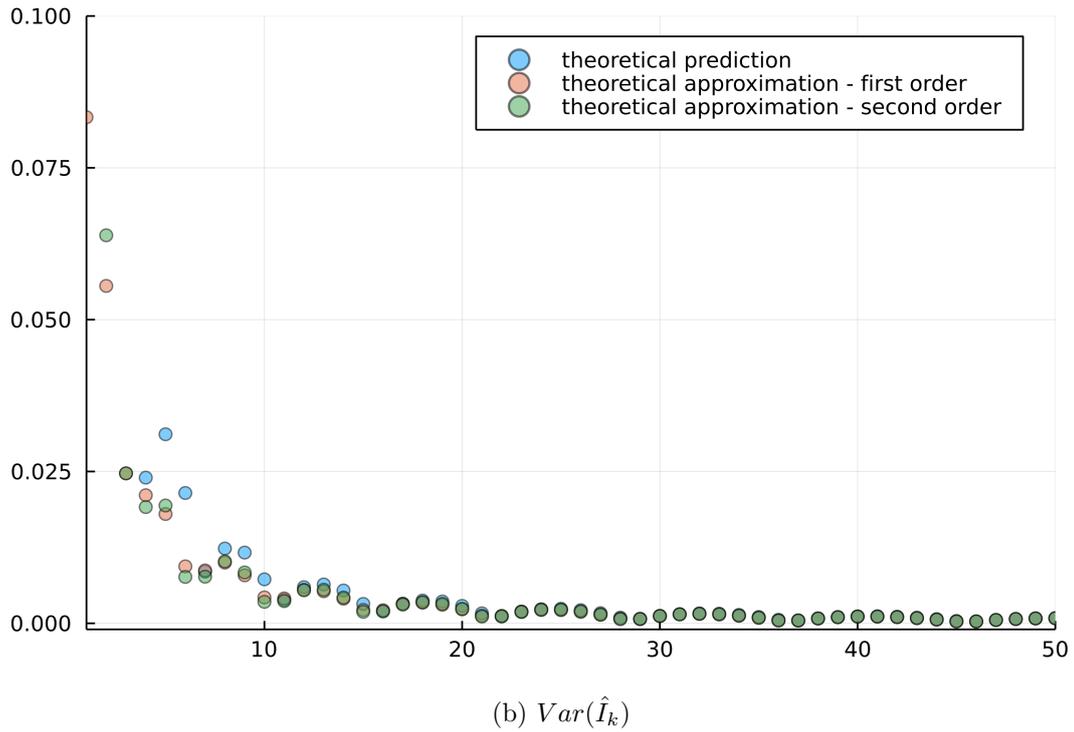


Figure 2.18.: Comparison of different the approximations

We can use our formula to search for some lower and upper bounds for the exception

value. We make use of the series representation of the hypergeometric function 1.14 to get an approximation to the function.

The bounds we found for  $k > 3$ :

- upper bound:  $1 + 1/x$
- lower bound:  $1 + 1/x^2$

We also plotted the average of the two bounds as some approximation. See 2.19, 2.20 and 2.21

At the moment of writing, it should be considered a conjecture. The idea of the proof is to look at the Taylor series approximation and use the remainder to find an upper bound and lower bound. A more formal proof is still needed.

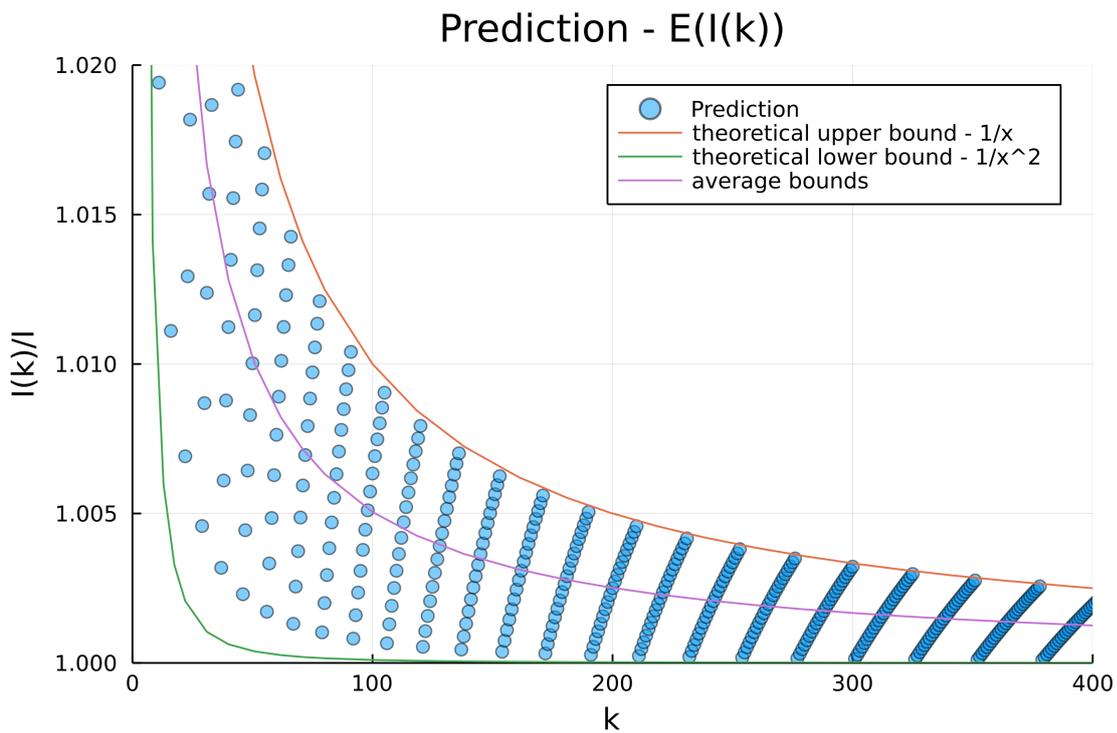


Figure 2.19.: Expectation value of stratification strategy 2

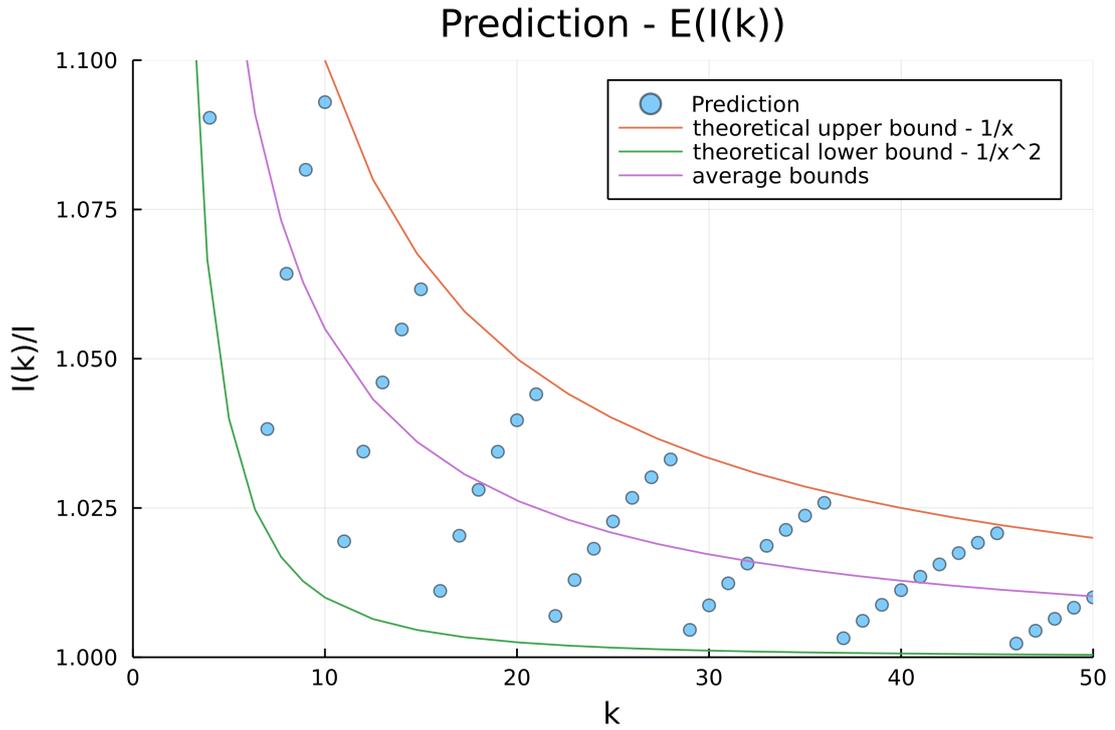


Figure 2.20.: Expectation value of stratification strategy 2 -  $k < 50$

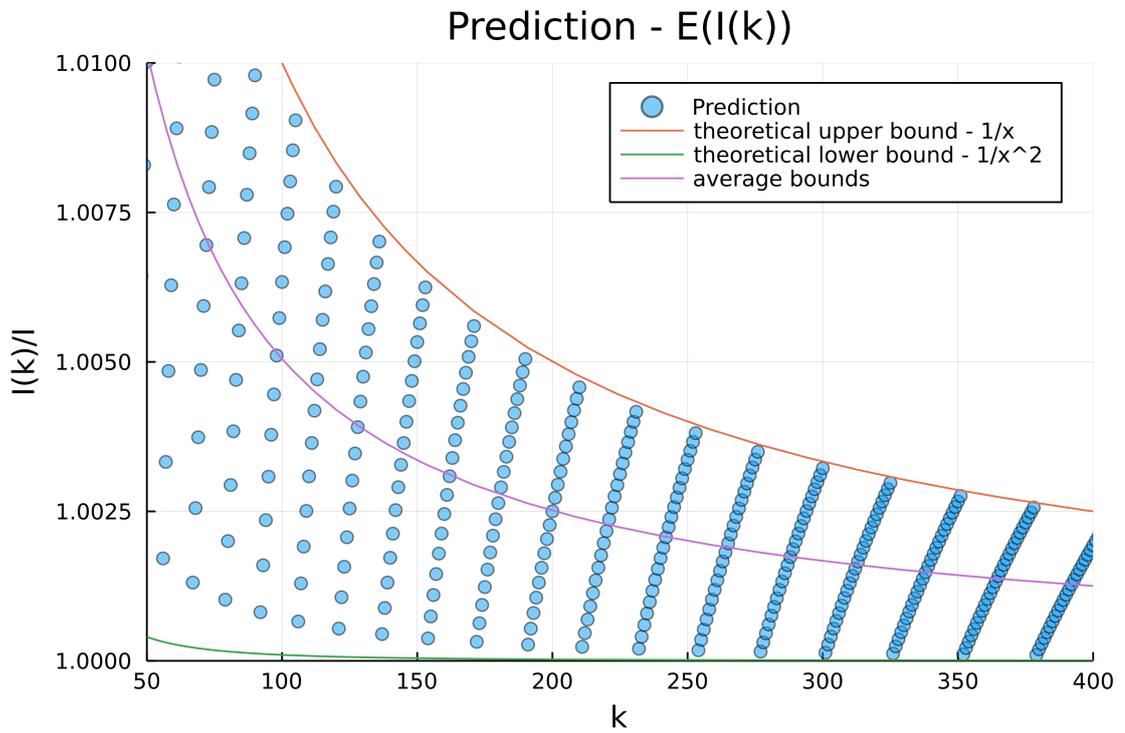


Figure 2.21.: Expectation value of stratification strategy 2 -  $k > 50$

## 3. Simulation

We will look in this part at the implementation of our simulation and compare the results to the theoretical predictions.

### 3.1. Program

I have written two different versions of the simulation.

#### 3.1.1. Version 1

I wrote a program to test the theoretical prediction against a simulation written in python. The experiment is based on the same model as in the previous chapter. We will emit photons onto a unit disc and look at the quantities at the center of the disc. I have used some scientific python packages to achieve fast results, like NumPy, SymPy, Matplotlib, and others. This model reached reasonably fast speeds but had a few minor errors, like not paying enough attention to index location ( $k=k-1$  location and  $\text{range}(k)$  did only go to  $k-1$ ), which were not easy to correct everywhere.

#### 3.1.2. Version 2

This version is an improved version of the previous program written in Julia<sup>1</sup> and makes use of the GPU. For the implementation of the hypergeometric function, I made use of a high-performance implementation of special mathematical functions in Julia<sup>2</sup>. The change to Julia and CUDA.jl [BFDS18] [BCEDS19]<sup>3</sup> makes on average a performance gain of factor five for most calculations. The CUDA implementation for the second stratification strategy did not gain as much performance as the first stratification strategy because I could not make clever use of the already partially sorted array. I am sure that the performance would be even better with more advanced techniques in CUDA programming. But for a simple implementation, I am pretty happy with this performance gain.

### 3.2. Optimizationa

To further increase the simulation speed, we implemented tricks that can only work for this setup. The optimizations do not affect the overall outcome of the simulation:

- We are working in spherical coordinates, which avoids rejection sampling. This choice comes particularly handy when the shells are pretty small, and rejection sampling would lead to poor performance. In addition, it already does calculate the distance for us.

---

<sup>1</sup><https://julialang.org/>

<sup>2</sup><https://github.com/JuliaMath/HypergeometricFunctions.jl>

<sup>3</sup><https://github.com/JuliaGPU/CUDA.jl>

### 3. Simulation

- We only generate photons in a shell, which enables us to sort the array quickly. Our list is always partially sorted, and only the last shell needs to be sorted before appending.
- For the final calculation, we only need the distance from our point p and not the coordinates, which allows us only to calculate the radius and not the angles. e.g., only one random number instead of 2 or 3

For generating the radii we used  $r = RAND (r_o - r_i) + r_i$ ,  $r = \sqrt{RAND (r_o^2 - r_i^2) + r_i^2}$  and  $r = \sqrt[3]{RAND (r_o^3 - r_i^3) + r_i^3}$  for the 1D, 2D, and 3D cases.

### 3.3. Simulation - Basic Results

For a simple sanity check, we are just plotting the results of one run of the code and repeat it a certain number of times. The check should give us an overview of whether the simulation behavior is as expected and does not make a mistake in the random numbers. We expect a lot of scattering for small k and less for bigger k. We can see that the first strategy does nicely scatter around our expected numbers while always staying above the real irradiance. See figure 3.1

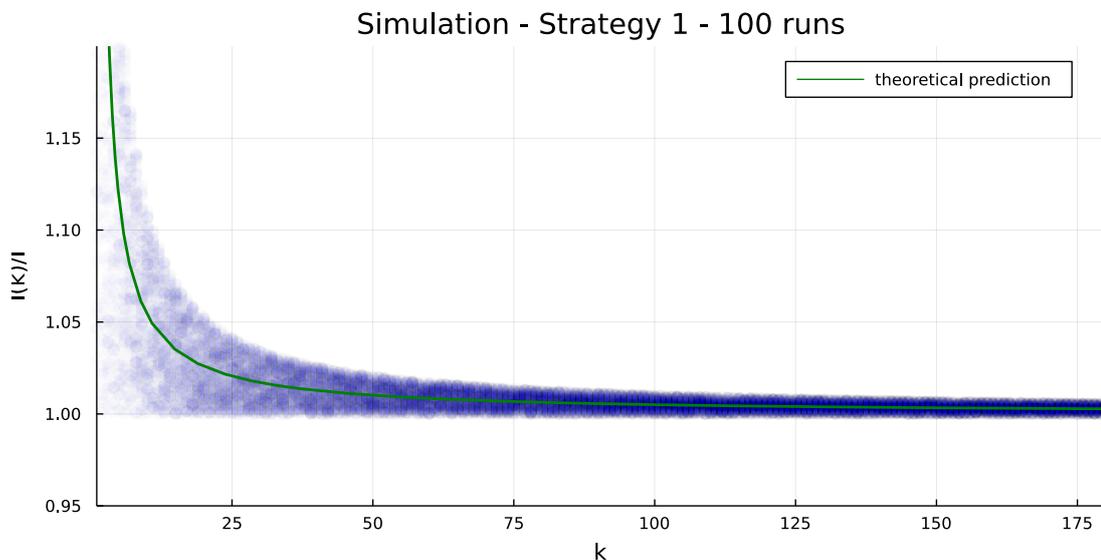


Figure 3.1.: Simulation - Strategy 1 - 100 runs

The second simulation shows quite a bit of randomness again. The samples are scattered below and above the real value. The two figures 3.2 3.3 show the results. Especial in figure 3.3 shows the pattern clearly.

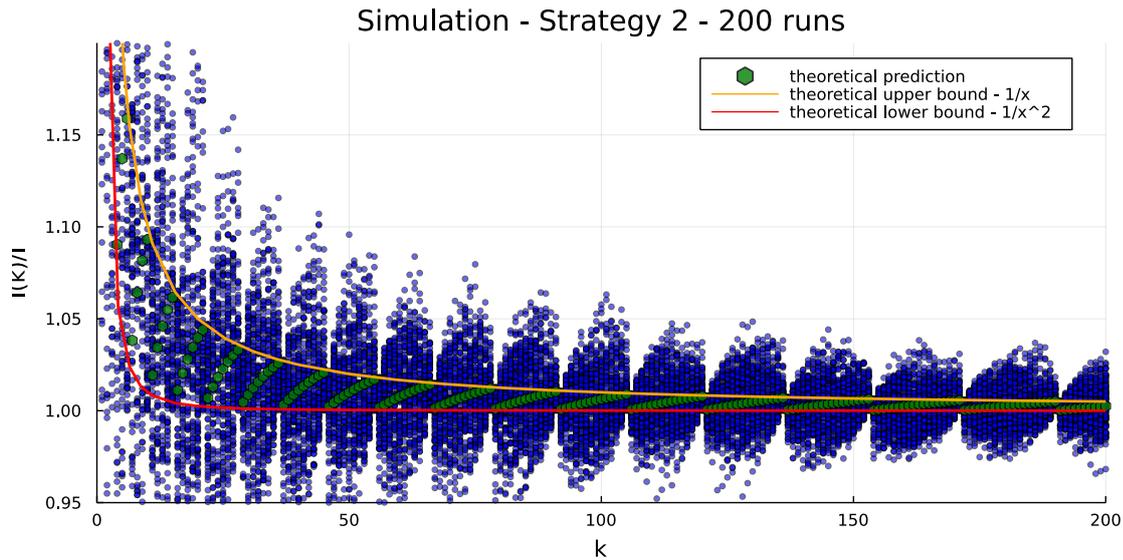


Figure 3.2.: Simulation - Strategy 2 - 200 runs

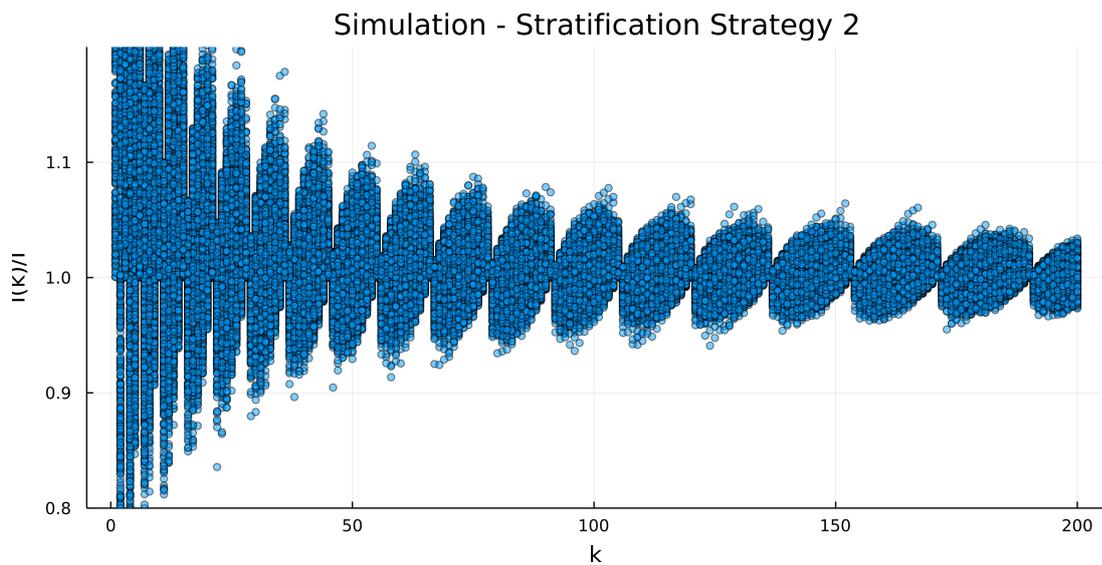


Figure 3.3.: Simulation - Strategy 2 - 1000 runs

The pattern is a consequence of the strategy we use to create the distribution of the photons. We can see that in contrast to the first strategy, we are getting some over- and underestimates. The pattern always has a similar shape for each of the shells. With some shrinking effect, the larger the  $k$ -values get.

Another sanity check is to look if we are getting better results the higher we choose the number of runs to average over. As expected, the results agree more with our predictions the higher the number of runs to average over (see figure 3.4). We can see that with 10 000 runs, the result is quite good and only has minimal noise for small  $k$ . For most examples, we

### 3. Simulation

choose, therefore, to average over 10 000 runs as a compromise between accuracy and time

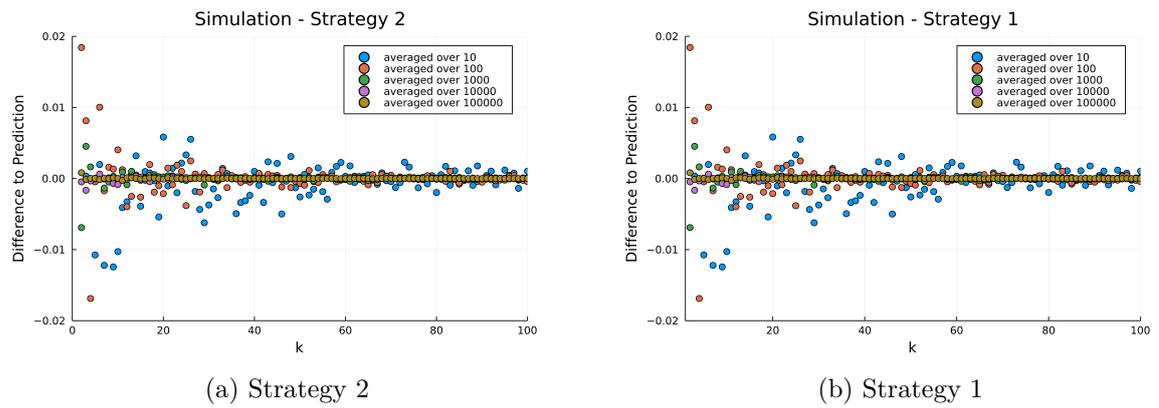


Figure 3.4.: Comparison of different number of reruns to average over

### 3.4. Simulation - Comparison Dimensions

This section will compare the results of the expectation values of our simulations in different dimensions. From our theoretical model, we expect to see that they are the same.

#### 3.4.1. Strategy 1

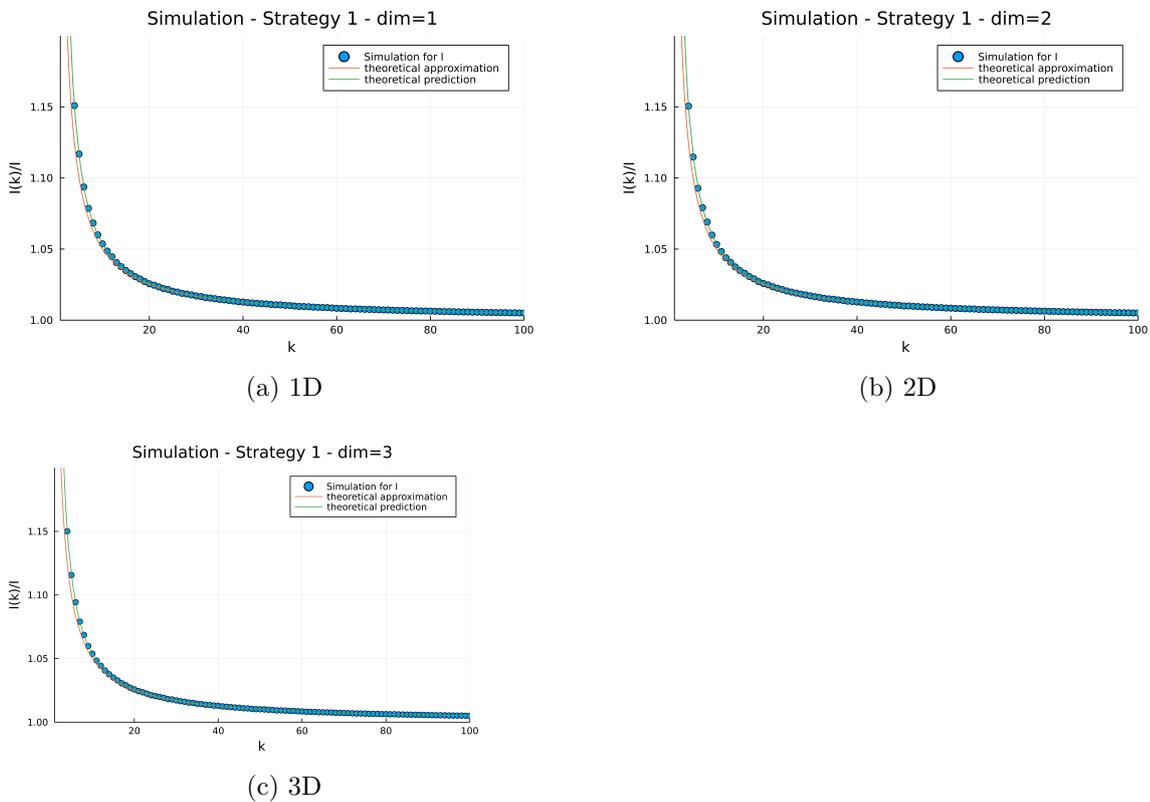


Figure 3.5.: All the simulation did run with 100 000 photons and were averaged over 10 000 runs

The results of the first stratification strategy agree to a reasonable degree of accuracy with our theoretical results (see 3.5). It shows that at least for our first stratification strategy, the results are independent of our dimension.

### 3. Simulation

#### 3.4.2. Strategy 2

For the second stratification strategy, we will repeat the simulation in different dimensions.

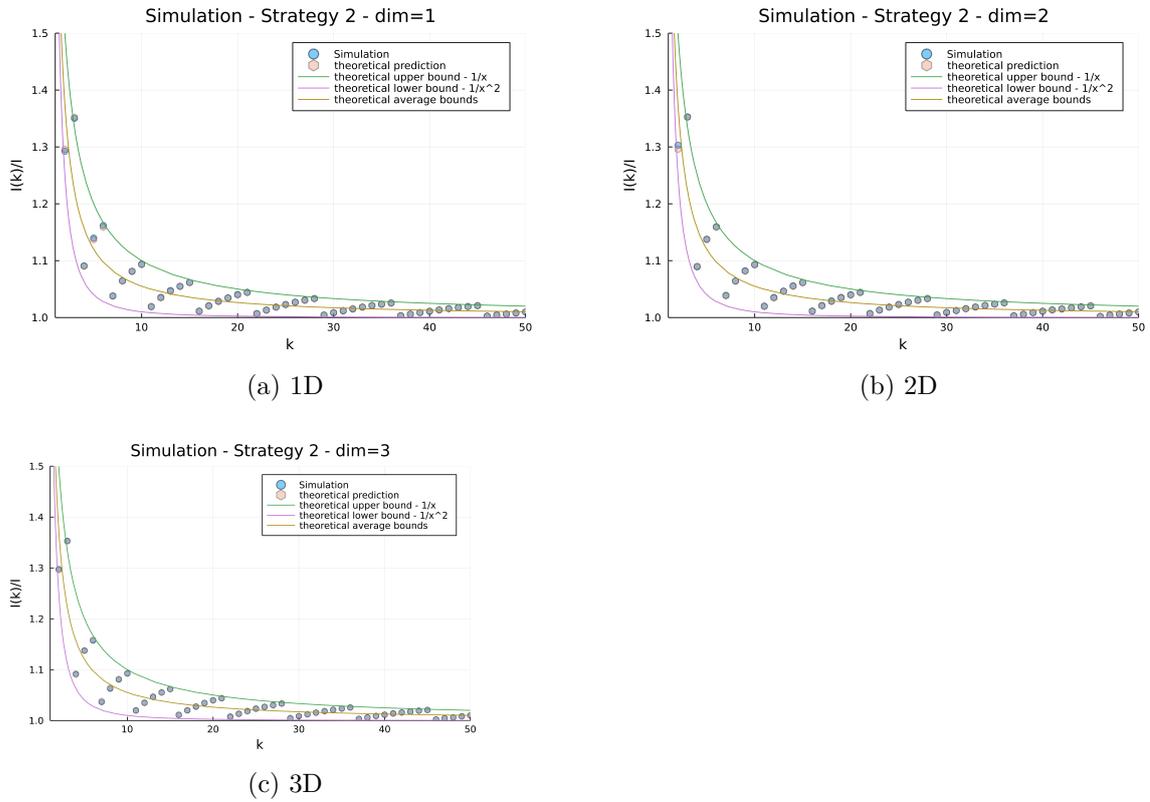


Figure 3.6.: All the simulation did run with 80 200 photons and were averaged over 10 000 runs

As expected, the result does not change if we change the dimensions of our simulation. Furthermore, we can see that that our theoretical predictions and empirical results are in good agreement. To simplify the further study, we will only carry the next simulations out in 2D only. This restriction should not affect the overall outcome.

### 3.5. Simulation – Expectation Value

This section now will compare the overall results between the prediction and simulation of the expectation value of the estimator. We will repeat the experiment for a different number of photons and will average this result over 10 000 runs.

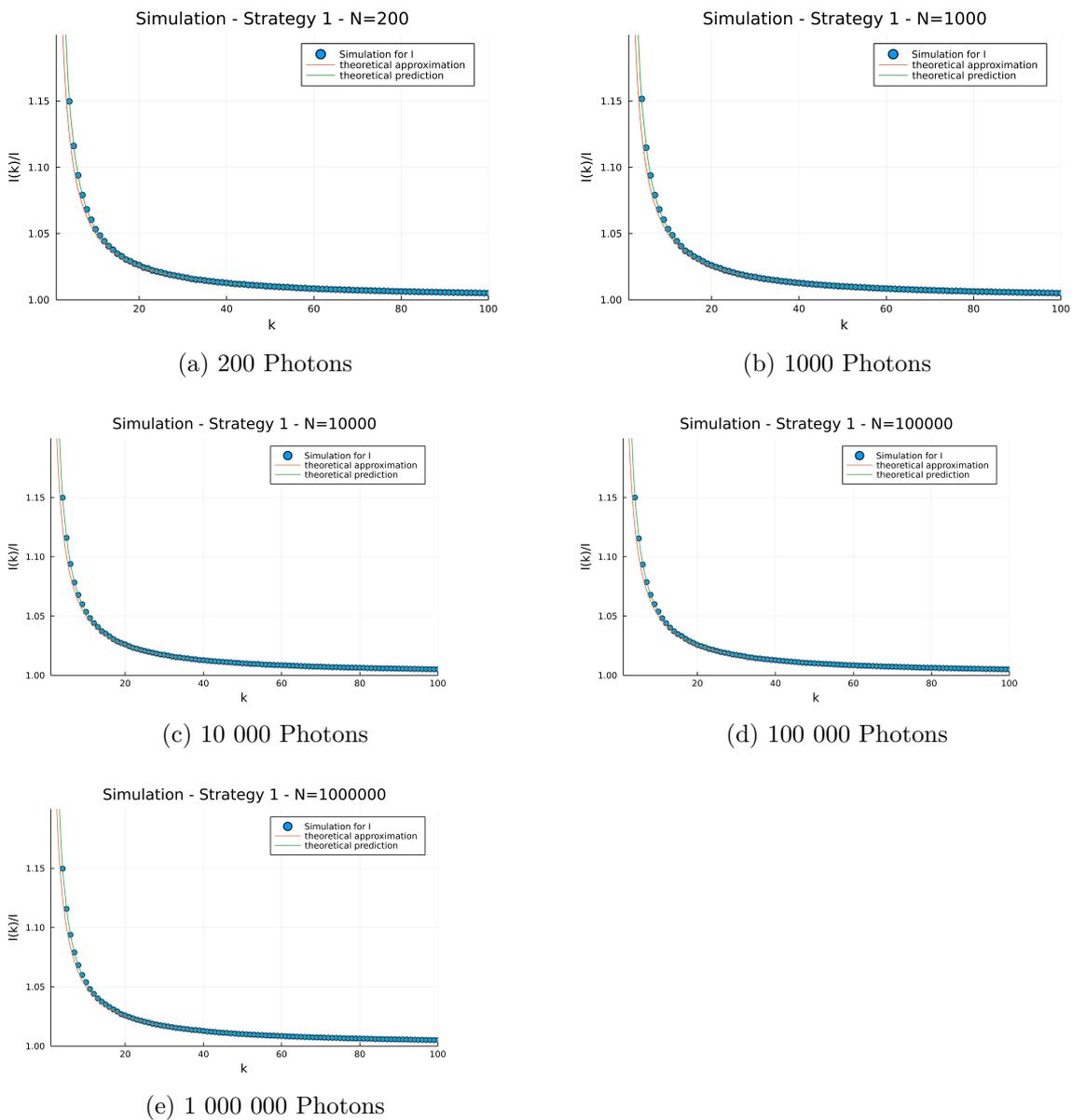


Figure 3.7.: All the simulation were averaged over 10 000 runs

### 3. Simulation

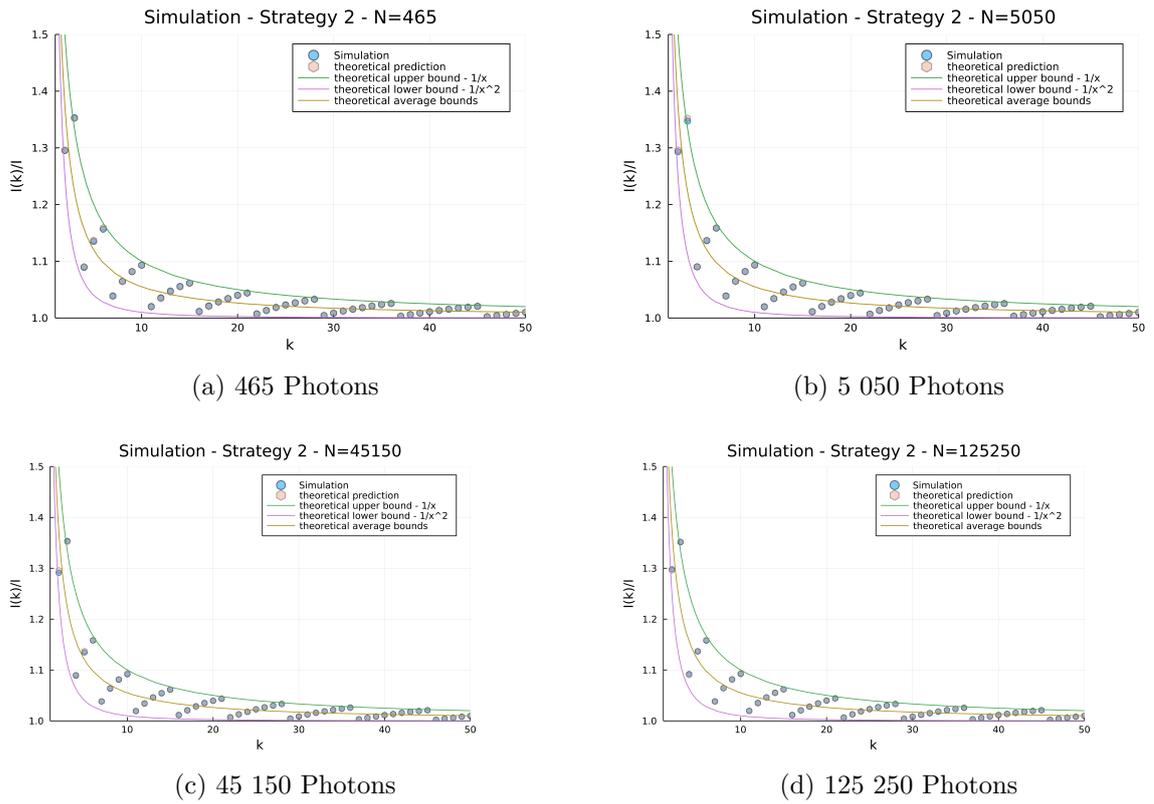


Figure 3.8.: All the simulation were averaged over 10 000 runs

The results of the simulation and the prediction are in good agreement. See figures 3.7 for strategy 1 and see figure 3.8 for strategy 2.

### 3.6. Simulation - Variance

In general, the variance is harder to calculate fast and exact. In the fast calculation I implemented, I was using  $Var(\hat{I}) = \sum_{runs} (I_{sim} - I_{Real})^2$  instead of  $Var(\hat{I}) = \sum_{runs} (I_{sim} - E(I_{sim}))^2$ , which resulted in a quite speedy calculation. Still, the faster version has a noticeable derivation from the prediction for smaller k. For a better accuracy I used DataFrames.jl<sup>4</sup> to store many runs and calculate the variance and mean based on this. We can see in figure 3.9 that as expected, the mean value does not change independently by calculated by the fast method or the method implemented in DataFrames.jl. While for the variance, we see a noticeable difference (see figure 3.9).

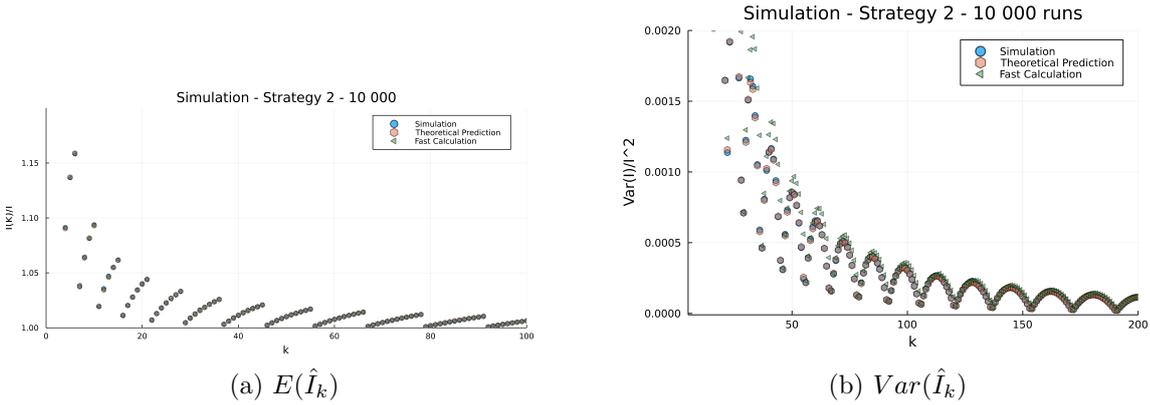


Figure 3.9.: Comparison of different methods in general

<sup>4</sup><https://github.com/JuliaData/DataFrames.jl>

### 3. Simulation

#### 3.6.1. Strategy 1

Since the simulation using strategy 1 is faster, which makes it easier to calculate the variance to good precision.

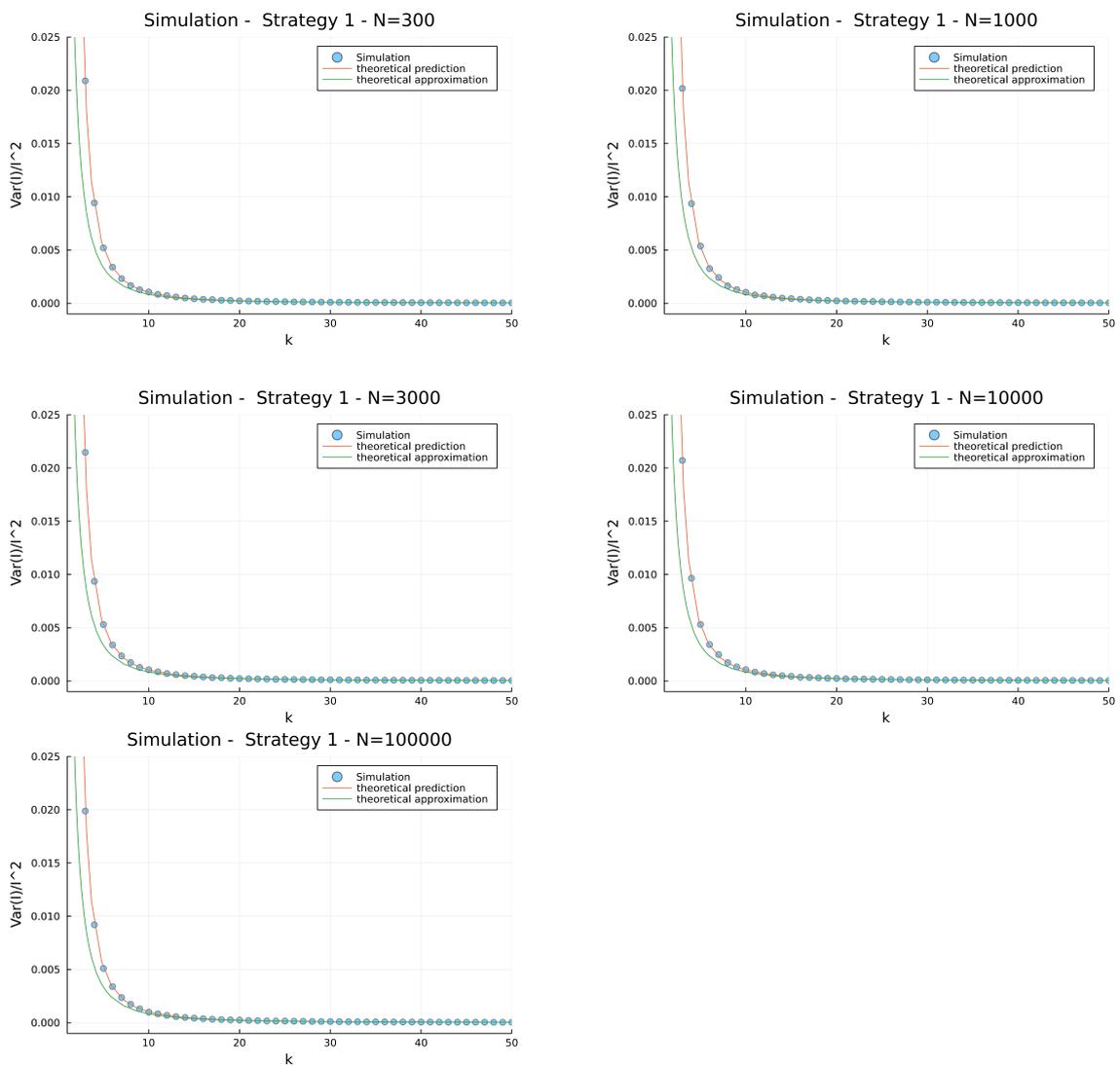


Figure 3.10.: All the simulation were averaged over 10 000 runs

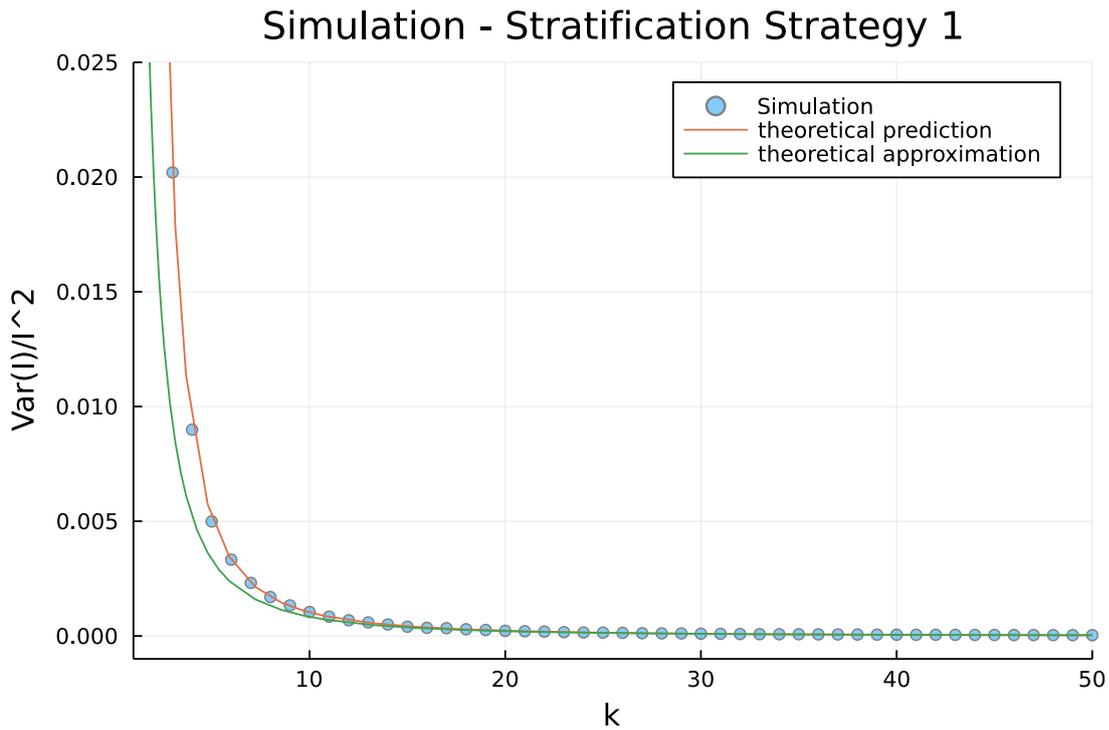


Figure 3.11.: Simulation - Strategy 1 - 1000 runs

### 3.6.2. Strategy 2

For strategy 2, the variance calculations are done in two ways: We can use a faster approach as explained in the beginning, which tends to differ by a lot for small values of k, and a slow but exacter approach to verify our result for smaller k.

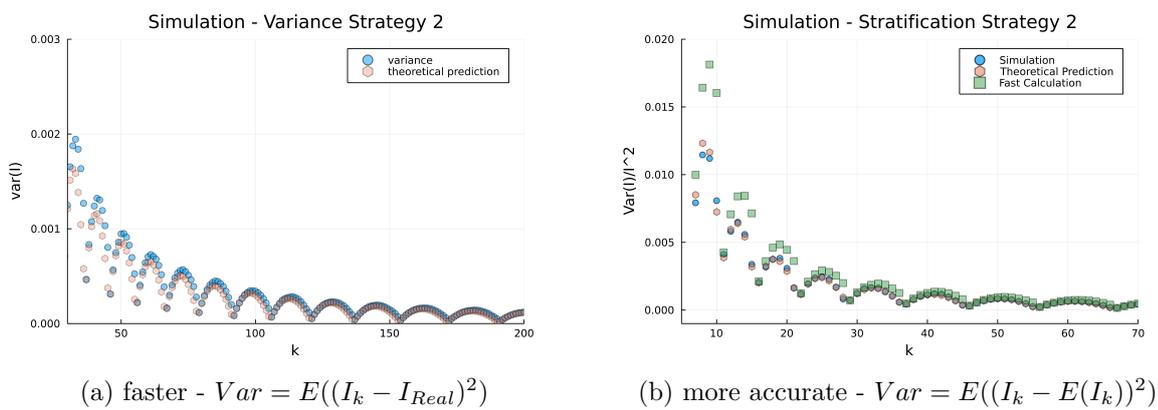


Figure 3.13.: Comparison of different methods to calculate Variance

3. Simulation

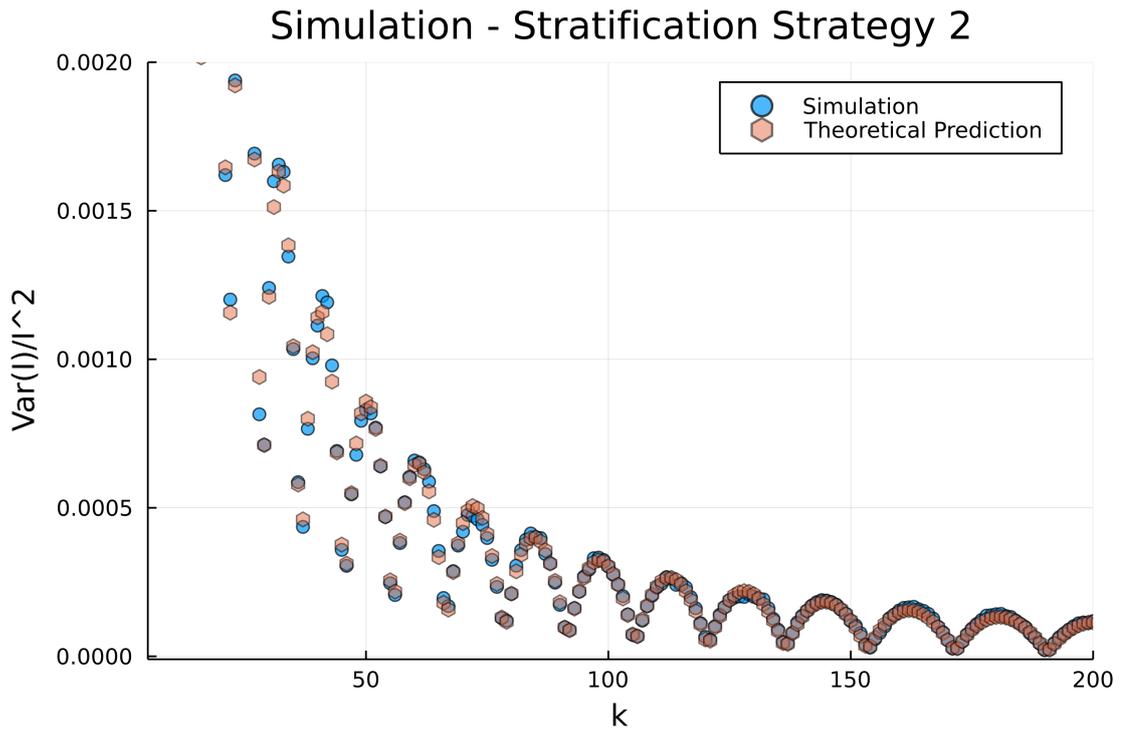


Figure 3.12.: Simulation - Strategy 2 - 1000 runs

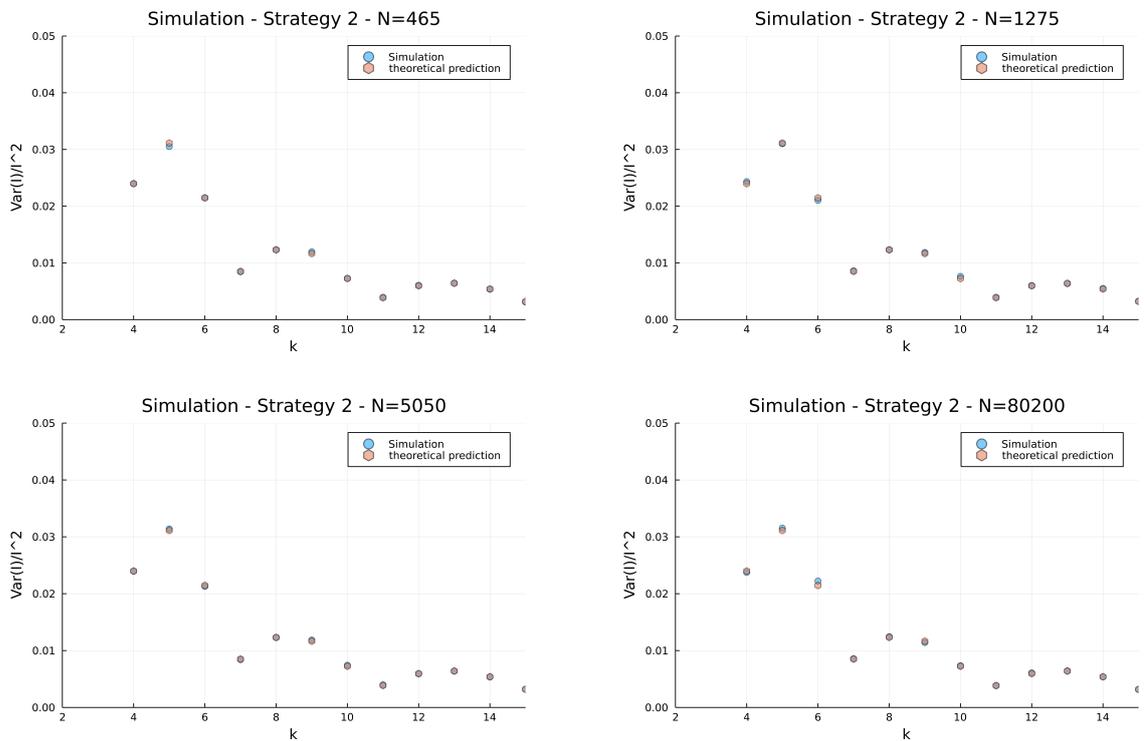


Figure 3.15.: All the simulation were averaged over 10 000 runs

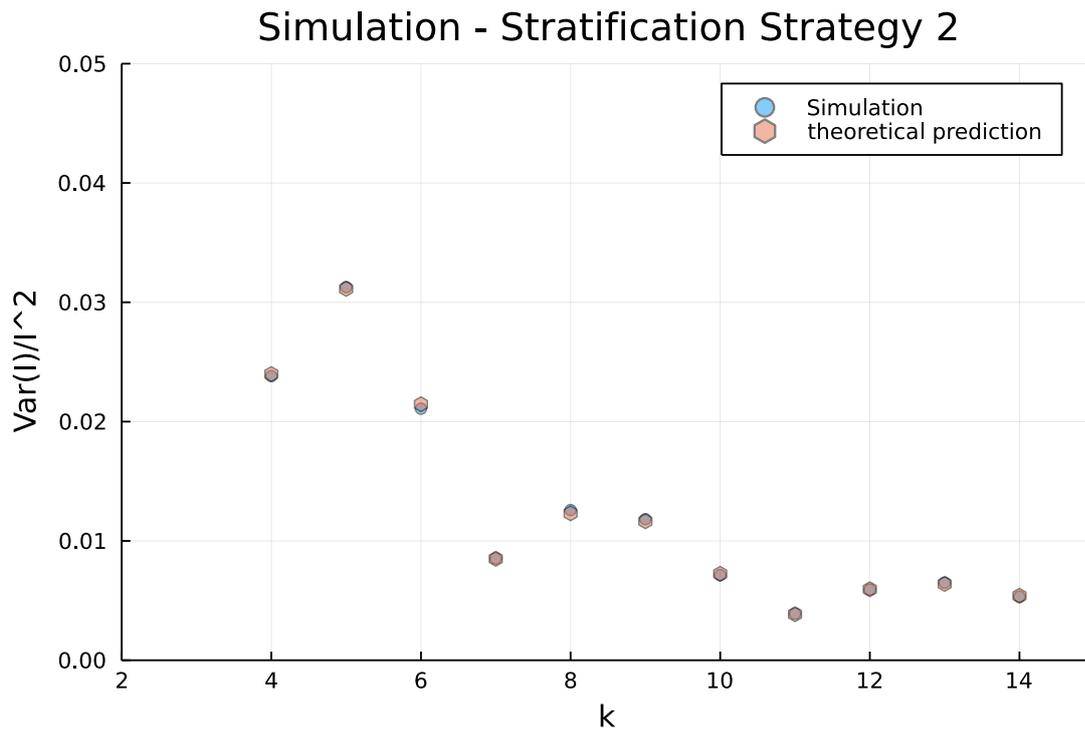


Figure 3.14.: Simulation - Strategy 2 - 10 000 runs



## 4. Conclusions and future work

This work has shown that stratification does reduce the variance of photon mapping in our particular setup. We outlined how one could deal with an arbitrary number of particles in a shell. An excellent next step would be to extend the stratification for a more general approach. Right now, we are only working in a particular framework that fully uses the spherical properties of our example and works only on one point. For rendering, we are mainly working pixels. For the most general case, we should use squares or other primitive geometry objects which can be arranged in a translation-invariant manner. This can be done by proving that our approach is a good enough approximation for  $k > k^{*1}$  or, in general, extending the framework to consider the border terms created by the not spherical forms. Another missing aspect is the study of kernels in the context of stratification. Another bachelor thesis is currently working on that. However, a step further would be to integrate the findings of this thesis to get approximations in the context of kernels.

---

<sup>1</sup> $k^*$  is here again a fixed but arbitrary number for which the relation holds



# A. Mathematical Appendix

We will summarize here a few critical mathematical functions and their properties. All those properties can be looked up in any mathematical handbook like [Bro99].

## A.1. Taylor Series

The Taylor series is a way to derive an approximation to differentiable functions within some boundaries. The general formula is:

$$f(x) = \sum_{i=0}^n \frac{f^{(i)}(0)}{i!} x^i + R_n \quad (\text{A.1})$$

While the rest part of the equation is given by:

$$R_n = \int_0^x \frac{(x-t)^n}{n!} f^{(n+1)}(t) dt \quad (\text{A.2})$$

We can use it to approximate the natural logarithm:

$$\text{Ln}(1+x) \sim x \quad (\text{A.3})$$

$$\frac{1}{1-x} \sim 1+x \quad (\text{A.4})$$

## A.2. Special Functions

We summarize here some basic properties of some crucial functions. For the logarithm, we have:

$$\text{Ln}(x) - \text{Ln}(y) = \text{Ln}\left(\frac{x}{y}\right) \quad (\text{A.5})$$

$$\text{Ln}(x) = \int \frac{1}{x} dx \quad (\text{A.6})$$

The gamma function properties:

$$\Gamma(n+1) = n\Gamma(n) \quad (\text{A.7})$$

$$\Gamma(a) = \int_0^\infty e^{-x} x^{a-1} dx \stackrel{\text{partial integration}}{=} (a-1) \int_0^\infty e^{-x} x^{a-2} dx \quad (\text{A.8})$$

$$= (a-1)\Gamma(a-1) \quad (\text{A.9})$$

## A. Mathematical Appendix

$$\Gamma(1/2) = \sqrt{\pi} \quad (\text{A.10})$$

The Beta function:

$$B(b, a) = B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \quad (\text{A.11})$$

Integral representation of the Beta function:

$$B(b, c-b) {}_2F_1(a, b; c; z) = \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a} dx \quad (\text{A.12})$$

$$R(c) > R(b) > 0 \quad (\text{A.13})$$

### A.3. Evaluation of Integral for Expectation Value

#### A.3.1. Evaluation of Integral for the 2d Case

Integrals of the form

$$I = \int_{V_a}^{V_b} dr C_{m,l} F^{l-1} (1-F)^{m-l} f \frac{1}{r^2} \quad (\text{A.14})$$

with

$$F = \frac{r^2 - V_a}{V_b - V_a} \quad (\text{A.15})$$

$f$  can be calculated by differentiation of  $F$  with respect to  $r$

$$f = \frac{2r}{V_b - V_a} \quad (\text{A.16})$$

Here  $V_a$  and  $V_b$  is the Volume/Area at the point a/b.

$$V \equiv V_b - V_a \quad (\text{A.17})$$

$$I = \int_{V_a}^{V_b} dr C \left(\frac{r^2 - V_a}{V}\right)^{l-1} \left(1 - \frac{r^2 - V_a}{V}\right)^{m-l} \frac{2}{rV} \quad (\text{A.18})$$

we introduce a variable transformation

$$r^2 = xV + V_a \quad (\text{A.19})$$

For the small changes in  $dr$  and  $dx$ , we get the following relation

$$2rdr = Vdx \quad (\text{A.20})$$

Applying simple manipulations to the integral

$$\int_{V_a}^{V_b} C \left(\frac{r^2 - V_a}{V}\right)^{l-1} \left(\frac{V - r^2 + V_a}{V}\right)^{m-l} \frac{1}{r^2V} 2rdr \quad (\text{A.21})$$

Using the coordinate transformation A.19 in equation A.21, the integral can be evaluated with the use of special functions.

$$J = \int_{V_a}^{V_b} dr I f_k(r) = \int_a^b C \left( \frac{r^2 - V_a}{V} \right)^{l-1} \left( \frac{V - r^2 + V_a}{V} \right)^{m-l} \frac{2r dr}{r^2 V} \quad (\text{A.22})$$

$$= \int_0^1 dx C(x)^{l-1} (1-x)^{m-l} \frac{1}{(xV + V_a)} \quad (\text{A.23})$$

$$= \int_0^1 dx C(x)^{l-1} (1-x)^{m-l} \left( x \frac{V}{V_a} + 1 \right)^{-1} \frac{1}{V_a} \quad (\text{A.24})$$

$$= \int_0^1 dx C(x)^{l-1} (1-x)^{m-l} \left( 1 - \left( -\frac{V}{V_a} \right) x \right)^{-1} \frac{1}{V_a} \quad (\text{A.25})$$

$$= \Gamma(l) \Gamma(-l + m + 1) {}_2\tilde{F}_1 \left( 1, l; m + 1; -\frac{v}{a} \right) \frac{1}{V_a} \quad (\text{A.26})$$

Here  $F$  denotes the hypergeometric regularized function, which according to [Bro99] is defined by:

$$B(b, c - b) {}_2F_1(a, b; c; z) = \int_0^1 x^{b-1} (1-x)^{c-b-1} (1-zx)^{-a} dx \quad (\text{A.27})$$

$$R(c) > R(b) > 0 \quad (\text{A.28})$$

### A.3.2. Evaluation of Integral of the general case

For the more general case:

$$F = \frac{r^d - V_a}{V_a - V_b} \quad (\text{A.29})$$

$$f = \frac{dr^{d-1}}{V} \quad (\text{A.30})$$

$$I = C \left( \frac{r^d - V_a}{V} \right)^{l-1} \left( 1 - \frac{r^d - V_a}{V} \right)^{m-l} \frac{d}{r} \quad (\text{A.31})$$

$$(\text{A.32})$$

The irradiance should scale like:

$$I r r \sim \frac{1}{r^d} \quad (\text{A.33})$$

where  $d$  is the dimension

using the same transformation as in the 2d case

$$r^d = xV + V_a \quad (\text{A.34})$$

For the transformation of  $dr$ , we get:

$$d r^{d-1} dr = V dx \quad (\text{A.35})$$

## A. Mathematical Appendix

With the help of the previous equations, the integral can be evaluated in the same manner as the last one

$$J = \int_a^b dr I = \int_a^b dr C \left( \frac{r^d - V_a}{V} \right)^{l-1} \left( \frac{V - r^d + V_a}{V} \right)^{m-l} \frac{d}{r} \frac{r^{d-1}}{r^{d-1}} \quad (\text{A.36})$$

$$= \int_0^1 dx C(x)^{l-1} (1-x)^{m-l} \frac{V}{(xV + V_a)} \frac{V}{V_a} \quad (\text{A.37})$$

$$= \int_0^1 dx C(x)^{l-1} (1-x)^{m-l} \left( x \frac{V}{V_a} + 1 \right)^{-1} \frac{V}{V_a} \quad (\text{A.38})$$

$$= \int_0^1 dx C(x)^{l-1} (1-x)^{m-l} \left( 1 - \left( -\frac{V}{V_a} \right) x \right)^{-1} \frac{V}{V_a} \quad (\text{A.39})$$

$$= \Gamma(l) \Gamma(-l + m + 1) {}_2\tilde{F}_1 \left( 1, l; m + 1; -\frac{v}{a} \right) \frac{V}{V_a} \quad (\text{A.40})$$

### A.4. Evaluation of Integral for the Variance

We can again write:

$$F = \frac{r^d - V_a}{V_a - V_b} \quad (\text{A.41})$$

$$f = \frac{dr^{d-1}}{V} \quad (\text{A.42})$$

$$I = \int_{V_a}^{V_b} dr \hat{C}_{m,l} F^{l-1} (1-F)^{m-l} f I^2 \quad (\text{A.43})$$

To evaluate the variance, we need to integrate over the squared irradiance:

$$I r r^2 \sim \frac{1}{r^{2d}} \quad (\text{A.44})$$

$$I = \int_{V_a}^{V_b} C \left( \frac{r^d - V_a}{V} \right)^{l-1} \left( 1 - \frac{r^d - V_a}{V} \right)^{m-l} \frac{d}{r^{2d}} r^{d-1} \quad (\text{A.45})$$

$$(\text{A.46})$$

using the same transformation as in the 2d case

$$r^d = xV + V_a \quad (\text{A.47})$$

For the transformation of  $dr$ , we get:

$$d r^{d-1} dr = V dx \quad (\text{A.48})$$

Using the previous equations, the integral evaluates in the same manner as the last one

$$J = \int_{V_a}^{V_b} dr I = \int_{V_a}^{V_b} dr C \left( \frac{r^d - V_a}{V} \right)^{l-1} \left( \frac{V - r^d + V_a}{V} \right)^{m-l} \frac{d}{r^{d+1}} \frac{r^{d-1}}{r^{d-1}} \quad (\text{A.49})$$

$$= \int_0^1 dx C (x)^{l-1} (1-x)^{m-l} \frac{V}{(xV + V_a)^2} \frac{V}{V_a} \quad (\text{A.50})$$

$$= \int_0^1 dx C (x)^{l-1} (1-x)^{m-l} \left( x \frac{V}{V_a} + 1 \right)^{-2} \frac{V}{V_a} \quad (\text{A.51})$$

$$= \int_0^1 dx C (x)^{l-1} (1-x)^{m-l} \left( 1 - \left( -\frac{V}{V_a} \right) x \right)^{-2} \frac{V}{V_a} \quad (\text{A.52})$$

$$= \Gamma(l) \Gamma(-l + m + 1) {}_2\tilde{F}_1 \left( 2, l; m + 1; -\frac{v}{a} \right) \frac{V}{V_a} \quad (\text{A.53})$$



## **B. Mathematica Code**

# Calculations for Photon Mapping

## Basic Definition

order statistic

$$\text{In[1]:= } \mathbf{fk}[F_, f_, n_, k_] := \frac{f n! F^{k-1} (1-F)^{n-k}}{(k-1)! (n-k)!}$$

$$\text{In[2]:= } \mathbf{fsimple}[F_, f_, c_] := f F^{1-1} (1-F)^{m-1} / r * c$$

$$\text{In[3]:= } \mathbf{factor}[n_, k_] := \frac{n!}{(k-1)! (n-k)!}$$

One dimensional irradiance

$$\text{In[4]:= } \mathbf{irr1d} = \frac{k J}{r n}$$

$$\text{Out[4]= } \frac{J k}{n r}$$

Two dimensional irradiance

$$\text{In[5]:= } \mathbf{irr2d} = \frac{k J}{n r^2}$$

$$\text{Out[5]= } \frac{J k}{n r^2}$$

$$\text{In[6]:= } \mathbf{irr3d} = \frac{k J}{n r^3}$$

$$\text{Out[6]= } \frac{J k}{n r^3}$$

Three dimensional "irradiance" Power Density

$$\text{Out[*]= } \frac{J k}{n r^3}$$

$$\text{In[7]:= } \mathbf{ei}[\mathbf{irr}_-, \mathbf{p}_] := \mathbf{irr} * \mathbf{p}$$

Expectation Value

$$\text{In[8]:= } \mathbf{EI}[\mathbf{ei}_-, \mathbf{a}_-, \mathbf{b}_] := \int_a^b \mathbf{ei} \, d\mathbf{r}$$

Variance

$$\text{In[9]:= } \mathbf{Var}[\mathbf{irr}_-, \mathbf{p}_-, \mathbf{a}_-, \mathbf{b}_] := \int_a^b \mathbf{irr}^2 * \mathbf{p} \, d\mathbf{r} - \left( \int_a^b \mathbf{irr} * \mathbf{p} \, d\mathbf{r} \right)^2$$

In[10]:=  $\$Assumptions = m - 1 > 0 \ \&\& \ l > 0 \ \&\& \ a > 0 \ \&\& \ v > 0 \ \&\& \ b > a \ \&\& \ a \in \text{Integers} \ \&\& \ b \in \text{Integers} \ \&\& \ n \in \text{Integers} \ \&\& \ k \in \text{Integers} \ \&\& \ l \in \text{Integers} \ \&\& \ m \in \text{Integers} \ \&\& \ k < n + 1 \ \&\& \ k > 0$

Out[10]=  $-1 + m > 0 \ \&\& \ l > 0 \ \&\& \ a > 0 \ \&\& \ v > 0 \ \&\& \ b > a \ \&\& \ a \in \mathbb{Z} \ \&\& \ b \in \mathbb{Z} \ \&\& \ n \in \mathbb{Z} \ \&\& \ k \in \mathbb{Z} \ \&\& \ l \in \mathbb{Z} \ \&\& \ m \in \mathbb{Z} \ \&\& \ k < 1 + n \ \&\& \ k > 0$

In[11]:=

## Order Statistic

### 1D

In[\*]:= **F1D0 = r**

Out[\*]= **r**

In[\*]:= **f1D0 = 1**

Out[\*]= **1**

In[\*]:= **p1D0 = fk[F1D0, f1D0, n, k]**

Out[\*]= 
$$\frac{(1 - r)^{-k+n} r^{-1+k} n!}{(-1 + k)! (-k + n)!}$$

In[\*]:= **ei1D0 = ei[irr1d, p1D0]**

Out[\*]= 
$$\frac{\int k (1 - r)^{-k+n} r^{-2+k} n!}{n (-1 + k)! (-k + n)!}$$

In[\*]:= **EI[ei1D0, 0, 1]**

Out[\*]= 
$$\frac{\int k \text{Gamma}[-1 + k]}{\text{Gamma}[k]} \text{ if } k > 1$$

In[\*]:= **FullSimplify[%]**

Out[\*]= 
$$\frac{\int k}{-1 + k} \text{ if } k > 1$$

In[\*]:= **Series**  $\left[ \frac{\int k}{-1 + k}, \{k, \text{Infinity}, 2\} \right]$

Out[\*]= 
$$\int + \frac{\int}{k} + \frac{\int}{k^2} + O\left[\frac{1}{k}\right]^3$$

In[\*]:= **Var[irr1d, p1D0, 0, 1]**

Out[\*]= 
$$-\frac{\int^2 k^2 \text{Gamma}[-1 + k]^2}{\text{Gamma}[k]^2} + \frac{\int^2 k^2 \text{Gamma}[-2 + k] \text{Gamma}[n]}{n \text{Gamma}[k] \text{Gamma}[-1 + n]} \text{ if } k > 2$$

In[\*]:= FullSimplify[%]

$$\text{Out[*]} = \frac{j^2 k^2 (1 - k + n)}{(-2 + k) (-1 + k)^2 n} \text{ if } k > 2$$

In[\*]:= Series  $\left[ \frac{j^2 k^2 (1 - k + n)}{(-2 + k) (-1 + k)^2 n}, \{n, \text{Infinity}, 1\}, \{k, \text{Infinity}, 1\} \right]$

$$\text{Out[*]} = \left( \frac{j^2}{k} + O\left[\frac{1}{k}\right]^2 \right) + \frac{-j^2 - \frac{3j^2}{k} + O\left[\frac{1}{k}\right]^2}{n} + O\left[\frac{1}{n}\right]^2$$

## 2D

In[\*]:= F2D0 = r^2

$$\text{Out[*]} = r^2$$

In[\*]:= f2D0 = 2 r

$$\text{Out[*]} = 2 r$$

In[\*]:= p2D0 = fk[F2D0, f2D0, n, k]

$$\text{Out[*]} = \frac{2 r (r^2)^{-1+k} (1 - r^2)^{-k+n} n!}{(-1 + k)! (-k + n)!}$$

In[\*]:= ei2D0 = ei[irr2d, p2D0]

$$\text{Out[*]} = \frac{2 j k (r^2)^{-1+k} (1 - r^2)^{-k+n} n!}{n r (-1 + k)! (-k + n)!}$$

In[\*]:= EI[ei2D0, 0, 1]

$$\text{Out[*]} = \frac{j k \text{Gamma}[-1 + k]}{\text{Gamma}[k]} \text{ if } k > 1$$

In[\*]:= FullSimplify[%]

$$\text{Out[*]} = \frac{j k}{-1 + k} \text{ if } k > 1$$

In[\*]:= Series  $\left[ \frac{j k}{-1 + k}, \{k, \text{Infinity}, 2\} \right]$

$$\text{Out[*]} = j + \frac{j}{k} + \frac{j}{k^2} + O\left[\frac{1}{k}\right]^3$$

In[\*]:= Var[irr2d, p2D0, 0, 1]

$$\text{Out[*]} = -\frac{j^2 k^2 \text{Gamma}[-1 + k]^2}{\text{Gamma}[k]^2} + \frac{j^2 k^2 \text{Gamma}[-2 + k] \text{Gamma}[n]}{n \text{Gamma}[k] \text{Gamma}[-1 + n]} \text{ if } k > 2$$

In[\*]:= FullSimplify[%]

$$\text{Out[*]} = \frac{j^2 k^2 (1 - k + n)}{(-2 + k) (-1 + k)^2 n} \text{ if } k > 2$$

In[\*]:= Series $\left[\frac{j^2 k^2 (1 - k + n)}{(-2 + k) (-1 + k)^2 n}, \{n, \text{Infinity}, 1\}, \{k, \text{Infinity}, 1\}\right]$

$$\text{Out[*]} = \left(\frac{j^2}{k} + O\left[\frac{1}{k}\right]^2\right) + \frac{-j^2 - \frac{3j^2}{k} + O\left[\frac{1}{k}\right]^2}{n} + O\left[\frac{1}{n}\right]^2$$

### 3D

In[\*]:= F3D0 = r^2

$$\text{Out[*]} = r^2$$

In[\*]:= f3D0 = 2 r

$$\text{Out[*]} = 2 r$$

In[\*]:= p3D0 = fk[F3D0, f3D0, n, k]

$$\text{Out[*]} = \frac{2 r (r^2)^{-1+k} (1 - r^2)^{-k+n} n!}{(-1 + k)! (-k + n)!}$$

In[\*]:= ei3D0 = ei[irr3d, p3D0]

$$\text{Out[*]} = \frac{2 j k (r^2)^{-2+k} (1 - r^2)^{-k+n} n!}{n (-1 + k)! (-k + n)!}$$

In[\*]:= EI[ei3D0, 0, 1]

$$\text{Out[*]} = \frac{j k \text{Gamma}\left[-\frac{3}{2} + k\right] \text{Gamma}[n]}{\text{Gamma}[k] \text{Gamma}\left[-\frac{1}{2} + n\right]} \text{ if } k > \frac{3}{2}$$

In[\*]:= FullSimplify $\left[\frac{j k \text{Gamma}[-1 + k] \text{Gamma}[n]}{\text{Gamma}[k] \text{Gamma}[n]}, n \in \text{Integers} \& k \in \text{Integers} \& n > k \& k > 2\right]$

$$\text{Out[*]} = \frac{j k}{-1 + k}$$

In[\*]:=

In[\*]:=

In[\*]:= Series $\left[\frac{j k}{-1 + k}, \{k, \text{Infinity}, 2\}\right]$

$$\text{Out[*]} = j + \frac{j}{k} + \frac{j}{k^2} + O\left[\frac{1}{k}\right]^3$$

In[\*]:= **Var**[irr3d, p3D0, 0, 1]

$$\text{Out[*]} = \frac{J^2 k^2 \text{Gamma}[-3+k] \text{Gamma}[n]}{n \text{Gamma}[k] \text{Gamma}[-2+n]} - \frac{J^2 k^2 \text{Gamma}\left[-\frac{3}{2}+k\right]^2 \text{Gamma}[n]^2}{\text{Gamma}[k]^2 \text{Gamma}\left[-\frac{1}{2}+n\right]^2} \text{ if } k > 3$$

In[\*]:= **FullSimplify** $\left[-\frac{J^2 k^2 \text{Gamma}\left[-\frac{3}{2}+k\right]^2 \text{Gamma}[n]^2}{\text{Gamma}[k]^2 \text{Gamma}\left[-\frac{1}{2}+n\right]^2} + \frac{J^2 k^2 n! \text{Gamma}[-3+k] \text{Gamma}[1-k+n]}{n^2 (-1+k)! (-k+n)! \text{Gamma}[-2+n]}\right]$

$$\text{Out[*]} = \frac{J^2 k^2 n! \left( \frac{\text{Gamma}[-3+k] \text{Gamma}[k]}{\text{Gamma}[-2+n]} - \frac{n! \text{Gamma}\left[-\frac{3}{2}+k\right]^2}{\text{Gamma}\left[-\frac{1}{2}+n\right]^2} \right)}{n^2 \text{Gamma}[k]^2}$$

In[\*]:= **Series** $\left[\frac{J^2 k^2 (1-k+n)}{(-2+k) (-1+k)^2 n}, \{n, \text{Infinity}, 1\}, \{k, \text{Infinity}, 1\}\right]$

$$\text{Out[*]} = \left( \frac{J^2}{k} + O\left[\frac{1}{k}\right]^2 \right) + \frac{-J^2 - \frac{3J^2}{k} + O\left[\frac{1}{k}\right]^2}{n} + O\left[\frac{1}{n}\right]^2$$

## Basic Stratification

### 1D

In[\*]:=

**start1DS = (k - 1) / (n)**

$$\text{Out[*]} = \frac{-1+k}{n}$$

In[\*]:= **end1DS = k / (n)**

$$\text{Out[*]} = \frac{k}{n}$$

In[\*]:= **F1DS = FullSimplify**[(r - a) / (b - a) /. {a → start1DS, b → end1DS}]

$$\text{Out[*]} = 1 - k + n r$$

In[\*]:= **f1DS = D**[F1DS, r]

$$\text{Out[*]} = n$$

In[\*]:= **p1DS = fk**[F1DS, f1DS, 1, 1]

$$\text{Out[*]} = n$$

In[\*]:= **ei1DS = ei**[irr1d, p1DS]

$$\text{Out[*]} = \frac{J k}{r}$$

In[\*]:= **EI**[ei1DS, start1DS, end1DS]

$$\text{Out[*]} = J k \text{Log} \left[ \frac{k}{-1+k} \right]$$

In[\*]:= **Series**  $\left[ J k \left( -\text{Log} \left[ \frac{-1+k}{n} \right] + \text{Log} \left[ \frac{k}{n} \right] \right), \{k, \text{Infinity}, 2\} \right]$

$$\text{Out[*]} = J + \frac{J}{2k} + \frac{J}{3k^2} + O \left[ \frac{1}{k} \right]^3$$

In[\*]:= **Var**[irr1d, p1DS, start1DS, end1DS]

$$\text{Out[*]} = \frac{J^2 k}{-1+k} - J^2 k^2 \text{Log} \left[ \frac{k}{-1+k} \right]^2$$

In[\*]:= **Series**  $\left[ \frac{J^2 k}{-1+k} - J^2 k^2 \left( -\text{Log} \left[ \frac{-1+k}{n} \right] + \text{Log} \left[ \frac{k}{n} \right] \right)^2, \{k, \text{Infinity}, 2\} \right]$

$$\text{Out[*]} = \frac{J^2}{12k^2} + O \left[ \frac{1}{k} \right]^3$$

## 2D

In[\*]:=

**start2DS = Sqrt** [(k - 1) / n]

$$\text{Out[*]} = \sqrt{\frac{-1+k}{n}}$$

In[\*]:= **end2DS = Sqrt** [k / n]

$$\text{Out[*]} = \sqrt{\frac{k}{n}}$$

In[\*]:= **F2DS = FullSimplify** [(r^2 - a^2) / (b^2 - a^2) /. {a → start2DS, b → end2DS}]

$$\text{Out[*]} = 1 - k + n r^2$$

In[\*]:= **f2DS = D**[F2DS, r]

$$\text{Out[*]} = 2 n r$$

In[\*]:= **p2DS = fk**[F2DS, f2DS, 1, 1]

$$\text{Out[*]} = 2 n r$$

In[\*]:= **ei2DS = ei**[irr2d, p2DS]

$$\text{Out[*]} = \frac{2 J k}{r}$$

In[\*]:= **EI**[ei2DS, start2DS, end2DS]

$$\text{Out[*]} = -J k \text{Log} \left[ \frac{-1+k}{k} \right]$$

`In[*]:= Series[J k (-Log[-1 + k/n] + Log[k/n]), {k, Infinity, 2}]`

`Out[*]:= J + J/(2 k) + J/(3 k^2) + O[1/k]^3`

`In[*]:= Var[irr2d, p2DS, start2DS, end2DS]`

`Out[*]:= J^2 k / (-1 + k) - J^2 k^2 Log[-1 + k/k]^2`

`In[*]:= Series[J^2 k / (-1 + k) - J^2 k^2 (-Log[-1 + k/n] + Log[k/n])^2, {k, Infinity, 2}]`

`Out[*]:= J^2 / (12 k^2) + O[1/k]^3`

### 3D

`In[*]:=`

`start3DS = (k - 1) / n ^ (1 / 3)`

`Out[*]:= (-1 + k / n) ^ (1 / 3)`

`In[*]:= end3DS = k / n ^ (1 / 3)`

`Out[*]:= (k / n) ^ (1 / 3)`

`In[*]:= F3DS = FullSimplify[(r^3 - a^3) / (b^3 - a^3) /. {a -> start3DS, b -> end3DS}]`

`Out[*]:= 1 - k + n r^3`

`In[*]:= f3DS = D[F3DS, r]`

`Out[*]:= 3 n r^2`

`In[*]:= p3DS = fk[F3DS, f3DS, 1, 1]`

`Out[*]:= 3 n r^2`

`In[*]:= ei3DS = ei[irr3d, p3DS]`

`Out[*]:= 3 J k / r`

`In[*]:= EI[ei3DS, start3DS, end3DS]`

`Out[*]:= -J k Log[-1 + k/k]`

`In[*]:= Series[J k (-Log[-1 + k/n] + Log[k/n]), {k, Infinity, 2}]`

`Out[*]:= J + J/(2 k) + J/(3 k^2) + O[1/k]^3`

In[\*]:= **Var**[irr3d, p3DS, start3DS, end3DS]

$$\text{Out[*]} = \frac{J^2 k}{-1+k} - J^2 k^2 \text{Log}\left[\frac{-1+k}{k}\right]^2$$

In[\*]:= **Series**\left[\frac{J^2 k}{-1+k} - J^2 k^2 \left(-\text{Log}\left[\frac{-1+k}{n}\right] + \text{Log}\left[\frac{k}{n}\right]\right)^2, \{k, \text{Infinity}, 2\}\right]

$$\text{Out[*]} = \frac{J^2}{12 k^2} + O\left[\frac{1}{k}\right]^3$$

## Order Statistic with Stratification

k is the lth photon in a shell with a total of m photons

generally the expectation value of l is

$\frac{v \Gamma[1] \Gamma[1-l+m] \text{Hypergeometric2F1Regularized}\left[1, 1, 1+m, -\frac{v}{va}\right]}{va}$  \*Constant with Va volume at point a and v the difference of Volume at point b minus Volume at point a

In[\*]:=

### 1D

In[\*]:= **Va1 = a^1**

Out[\*]= **a**

In[\*]:= **Vb1 = b^1**

Out[\*]= **b**

In[\*]:= **dv1 = Vb1 - Va1**

Out[\*]= **-a + b**

In[\*]:=

In[\*]:= **F1 = (r - Va1) / (Vb1 - Va1)**

Out[\*]=  $\frac{-a+r}{-a+b}$

In[\*]:= **Dr F1**

Out[\*]=  $\frac{1}{-a+b}$

In[\*]:= **f1 = D[F1, r]**

Out[\*]=  $\frac{1}{-a+b}$

In[\*]:= **eqos1 = fk[F1, f1, m, l] \* irr1d**

Out[\*]= 
$$\frac{J k \left(\frac{-a+r}{-a+b}\right)^{-1+l} \left(1 - \frac{-a+r}{-a+b}\right)^{-1+m} m!}{(-a+b) n r (-1+l)! (-1+m)!}$$

Integral transformation (see appendix)

In[\*]:= Integrate[x^(1-1) \* (1-x)^(1-1) \* (x\*dv1+Va1)^-1, {x, 0, 1}] \* dv1 \* factor[k, n]

$$\text{Out[*]} = \frac{k! \operatorname{Log}\left[\frac{b}{a}\right]}{(k-n)! (-1+n)!}$$

In[\*]:= 
$$\frac{(-a+b) k! (-1+m)! \operatorname{Gamma}[1] \operatorname{Hypergeometric2F1Regularized}\left[1, 1, 1+m, 1-\frac{b}{a}\right]}{a (k-n)! (-1+n)!} /.$$

{1 → 1, m → 1}

$$\text{Out[*]} = -\frac{(-a+b) k! \operatorname{Log}\left[\frac{b}{a}\right]}{(a-b) (k-n)! (-1+n)!}$$

In[\*]:= Integrate[x^(1-1) \* (1-x)^(m-1) \* (x\*dv1+Va1)^-1, {x, 0, 1}] \* dv1 \* factor[k, n]

$$\text{Out[*]} = \frac{(-a+b) k! (-1+m)! \operatorname{Gamma}[1] \operatorname{Hypergeometric2F1Regularized}\left[1, 1, 1+m, 1-\frac{b}{a}\right]}{a (k-n)! (-1+n)!}$$

In[\*]:= EIos1 =

$$\frac{(-a+b) k! \operatorname{Gamma}[1] \operatorname{Gamma}[1-1+m] \operatorname{Hypergeometric2F1Regularized}\left[1, 1, 1+m, 1-\frac{b}{a}\right]}{a (k-n)! (-1+n)!} *$$

J \* k / n

$$\text{Out[*]} = \frac{(-a+b) J k k! \operatorname{Gamma}[1] \operatorname{Gamma}[1-1+m] \operatorname{Hypergeometric2F1Regularized}\left[1, 1, 1+m, 1-\frac{b}{a}\right]}{a n (k-n)! (-1+n)!}$$

integrate I^2

In[\*]:= Integrate[x^(1-1) \* (1-x)^(m-1) \* (x\*dv1+Va1)^-2, {x, 0, 1}] \* dv1 \* factor[k, n]

$$\text{Out[*]} = \frac{(-a+b) k! (-1+m)! \operatorname{Gamma}[1] \left( a m + (b-b l + a (1-m)) \operatorname{Hypergeometric2F1}\left[1, 1, 1+m, 1-\frac{b}{a}\right] \right)}{a^2 b m! (k-n)! (-1+n)!}$$

In[\*]:= EI2os1 = 
$$\frac{1}{a^2 b (k-n)! (-1+n)!} (-a+b) k! \operatorname{Gamma}[1] \operatorname{Gamma}[1-1+m]$$

$$\left( \frac{a}{\operatorname{Gamma}[m]} + (b-b l + a (1-m)) \operatorname{Hypergeometric2F1Regularized}\left[1, 1, 1+m, 1-\frac{b}{a}\right] \right) * (J * k / n)^2$$

$$\text{Out[*]} = \left( (-a+b) J^2 k^2 k! \operatorname{Gamma}[1] \operatorname{Gamma}[1-1+m] \left( \frac{a}{\operatorname{Gamma}[m]} + (b-b l + a (1-m)) \operatorname{Hypergeometric2F1Regularized}\left[1, 1, 1+m, 1-\frac{b}{a}\right] \right) \right) / (a^2 b n^2 (k-n)! (-1+n)!)$$

In[ ]:= **Varos1 = Simplify[EI2os1 - EIos1^2]**

$$\text{Out[ ]} = -\frac{1}{a^2 b n^2 ((k-n)!)^2 ((-1+n)!)^2 \text{Gamma}[m]} (a-b) J^2 k^2 k! \text{Gamma}[1] \text{Gamma}[1-l+m] \left( (a-b) b k! \text{Gamma}[1] \text{Gamma}[m] \text{Gamma}[1-l+m] \right. \\ \left. \text{Hypergeometric2F1Regularized}\left[1, 1, 1+m, 1-\frac{b}{a}\right]^2 + (k-n)! (-1+n)! \right. \\ \left. \left( a + (b-b l + a(1-m)) \text{Gamma}[m] \text{Hypergeometric2F1Regularized}\left[1, 1, 1+m, 1-\frac{b}{a}\right] \right) \right)$$

## 2D

In[ ]:= **Va2 = a^2**

Out[ ]:=  $a^2$

In[ ]:= **Vb2 = b^2**

Out[ ]:=  $b^2$

In[ ]:= **dv2 = Vb2 - Va2**

Out[ ]:=  $-a^2 + b^2$

In[ ]:= **F2 = (r^2 - Va2) / (Vb2 - Va2)**

Out[ ]:=  $\frac{-a^2 + r^2}{-a^2 + b^2}$

In[ ]:= **f2 = D[F2, r]**

Out[ ]:=  $\frac{2 r}{-a^2 + b^2}$

In[ ]:= **eqos2 = fk[F2, f2, m, 1] \* irr2d**

Out[ ]:=  $\frac{2 J k \left( \frac{-a^2+r^2}{-a^2+b^2} \right)^{-1+1} \left( 1 - \frac{-a^2+r^2}{-a^2+b^2} \right)^{-1+m} m!}{(-a^2 + b^2) n r (-1+1)! (-1+m)!}$

Integral transformation (see appendix)

**Integrate[x^(1-1) \* (1-x)^(m-1) \* (x \* dv2 + Va2)^-1, {x, 0, 1}] \* dv2**

Out[ ]:=  $\frac{(-a^2 + b^2) k! (-1+m)! \text{Gamma}[1] \text{Hypergeometric2F1Regularized}\left[1, 1, 1+m, 1-\frac{b^2}{a^2}\right]}{a^2 (k-n)! (-1+n)!}$

**EIos2 =**

$$\frac{(-a^2 + b^2) k! \text{Gamma}[1] \text{Gamma}[1-l+m] \text{Hypergeometric2F1Regularized}\left[1, 1, 1+m, 1-\frac{b^2}{a^2}\right]}{a^2 (n-k)! (-1+n)!} * \\ J * k / n \\ \text{Out[ ]} = \frac{(-a^2 + b^2) J k k! \text{Gamma}[1] \text{Gamma}[1-l+m] \text{Hypergeometric2F1Regularized}\left[1, 1, 1+m, 1-\frac{b^2}{a^2}\right]}{a^2 n (k-n)! (-1+n)!}$$

$$\text{In}[*]:= \frac{(-a^2 + b^2) k! \text{Gamma}[1] \text{Gamma}[1 - 1 + m] \text{Hypergeometric2F1Regularized}\left[1, 1, 1 + m, 1 - \frac{b^2}{a^2}\right]}{a^2 (k - n)! (-1 + n)!} /.$$

{1 → 1, m → 1}

$$\text{Out}[*]:= -\frac{(-a^2 + b^2) k! \text{Log}\left[\frac{b^2}{a^2}\right]}{a^2 \left(1 - \frac{b^2}{a^2}\right) (k - n)! (-1 + n)!}$$

integrate I^2

$$\text{In}[*]:= \text{Integrate}[x^{(1-1)} * (1-x)^{(m-1)} * (x * dv2 + Va2)^{-2}, \{x, \theta, 1\}] * dv2 * \text{factor}[k, n]$$

$$\text{Out}[*]:= \left( (-a^2 + b^2) k! (-1 + m)! \text{Gamma}[1] \right. \\ \left. \left( a^2 m + (-b^2 (-1 + 1) + a^2 (1 - m)) \text{Hypergeometric2F1}\left[1, 1, 1 + m, 1 - \frac{b^2}{a^2}\right] \right) \right) / \\ (a^4 b^2 m! (k - n)! (-1 + n)!)$$

$$\text{In}[*]:= \text{EI2os2} = \frac{1}{a^4 b^2 (k - n)! (-1 + n)!} (-a^2 + b^2) k! \text{Gamma}[1] \text{Gamma}[1 - 1 + m]$$

$$\left( \frac{a^2}{\text{Gamma}[m]} + (-b^2 (-1 + 1) + a^2 (1 - m)) \text{Hypergeometric2F1Regularized}\left[1, 1, 1 + m, 1 - \frac{b^2}{a^2}\right] \right) * \\ (J * k / n)^2$$

$$\text{Out}[*]:= \left( (-a^2 + b^2) J^2 k^2 k! \text{Gamma}[1] \text{Gamma}[1 - 1 + m] \left( \frac{a^2}{\text{Gamma}[m]} + (-b^2 (-1 + 1) + a^2 (1 - m)) \right. \right. \\ \left. \left. \text{Hypergeometric2F1Regularized}\left[1, 1, 1 + m, 1 - \frac{b^2}{a^2}\right] \right) \right) / (a^4 b^2 n^2 (k - n)! (-1 + n)!)$$

$$\text{In}[*]:= \text{Varos2} = \text{Simplify}[\text{EI2os2} - \text{EIos2}^2]$$

$$\text{Out}[*]:= -\frac{1}{a^4 b^2 n^2 ((k - n)!)^2 ((-1 + n)!)^2 \text{Gamma}[m]} \\ (a^2 - b^2) J^2 k^2 k! \text{Gamma}[1] \text{Gamma}[1 - 1 + m] \left( b^2 (a^2 - b^2) k! \text{Gamma}[1] \text{Gamma}[m] \right. \\ \left. \text{Gamma}[1 - 1 + m] \text{Hypergeometric2F1Regularized}\left[1, 1, 1 + m, 1 - \frac{b^2}{a^2}\right]^2 + \right. \\ \left. (k - n)! (-1 + n)! \left( a^2 + (-b^2 (-1 + 1) + a^2 (1 - m)) \text{Gamma}[m] \right. \right. \\ \left. \left. \text{Hypergeometric2F1Regularized}\left[1, 1, 1 + m, 1 - \frac{b^2}{a^2}\right] \right) \right)$$

### 3D

$$\text{In}[*]:= \text{Va3} = a^3$$

$$\text{Out}[*]:= a^3$$

In[\*]:= **Vb3 = b^3**

Out[\*]=  $b^3$

In[\*]:= **dv3 = Vb3 - Va3**

Out[\*]=  $-a^3 + b^3$

In[\*]:= **F3 = (r^3 - Va3) / (Vb3 - Va3)**

Out[\*]= 
$$\frac{-a^3 + r^3}{-a^3 + b^3}$$

In[\*]:= **f3 = D[F3, r]**

Out[\*]= 
$$\frac{3 r^2}{-a^3 + b^3}$$

In[\*]:= **fk[F3, f3, m, l] \* irr3d**

Out[\*]= 
$$\frac{3 J k \left( \frac{-a^3+r^3}{-a^3+b^3} \right)^{-1+l} \left( 1 - \frac{-a^3+r^3}{-a^3+b^3} \right)^{-1+m} m!}{(-a^3 + b^3) n r (-1 + l)! (-1 + m)!}$$

In[\*]:= **eqos3 = fk[F3, f3, m, l] \* irr3d**

Out[\*]= 
$$\frac{3 J k \left( \frac{-a^3+r^3}{-a^3+b^3} \right)^{-1+l} \left( 1 - \frac{-a^3+r^3}{-a^3+b^3} \right)^{-1+m} m!}{(-a^3 + b^3) n r (-1 + l)! (-1 + m)!}$$

Integral transformation (see appendix)

In[\*]:= **Integrate[x^(l-1) \* (1-x)^(m-1) \* (x \* dv3 + Va3)^-1, {x, 0, 1}] \* dv3 \* factor[k, n]**

Out[\*]= 
$$\frac{(-a^3 + b^3) k! (-1 + m)! \Gamma[1] \text{Hypergeometric2F1Regularized}\left[1, 1, 1 + m, 1 - \frac{b^3}{a^3}\right]}{a^3 (k - n)! (-1 + n)!}$$

In[\*]:= **EIos3 =**

Out[\*]= 
$$\frac{(-a^3 + b^3) k! \Gamma[1] \Gamma[1 - l + m] \text{Hypergeometric2F1Regularized}\left[1, 1, 1 + m, 1 - \frac{b^3}{a^3}\right]}{a^3 (k - n)! (-1 + n)!} * J * k / n$$

Out[\*]= 
$$\frac{(-a^3 + b^3) J k k! \Gamma[1] \Gamma[1 - l + m] \text{Hypergeometric2F1Regularized}\left[1, 1, 1 + m, 1 - \frac{b^3}{a^3}\right]}{a^3 n (k - n)! (-1 + n)!}$$

integrate l^2

In[\*]:= **Integrate[x^(l-1) \* (1-x)^(m-1) \* (x \* dv3 + Va3)^-2, {x, 0, 1}] \* dv3 \* factor[k, n]**

Out[\*]= 
$$\left( (-a^3 + b^3) k! (-1 + m)! \Gamma[1] \left( a^3 m + (-b^3 (-1 + l) + a^3 (1 - m)) \text{Hypergeometric2F1}\left[1, 1, 1 + m, 1 - \frac{b^3}{a^3}\right] \right) \right) / (a^6 b^3 m! (k - n)! (-1 + n)!)$$

$$\text{In[*]:= EI2os3} = \frac{1}{a^6 b^3 (k-n)! (-1+n)!} (-a^3 + b^3) k! \text{Gamma}[1] \text{Gamma}[1-1+m]$$

$$\left( \frac{a^3}{\text{Gamma}[m]} + (-b^3 (-1+1) + a^3 (1-m)) \text{Hypergeometric2F1Regularized}\left[1, 1, 1+m, 1 - \frac{b^3}{a^3}\right] \right)$$

$$(J * k / n)^2$$

$$\text{Out[*]:=} \left( (-a^3 + b^3) J^2 k^2 k! \text{Gamma}[1] \text{Gamma}[1-1+m] \left( \frac{a^3}{\text{Gamma}[m]} + (-b^3 (-1+1) + a^3 (1-m)) \right) \right.$$

$$\left. \text{Hypergeometric2F1Regularized}\left[1, 1, 1+m, 1 - \frac{b^3}{a^3}\right] \right) / (a^6 b^3 n^2 (k-n)! (-1+n)!)$$

**Varos3 = Simplify[EI2os3 - EIos3^2]**

$$\text{Out[*]:=} - \frac{1}{a^6 b^3 n^2 ((k-n)!)^2 ((-1+n)!)^2 \text{Gamma}[m]}$$

$$(a^3 - b^3) J^2 k^2 k! \text{Gamma}[1] \text{Gamma}[1-1+m] \left( b^3 (a^3 - b^3) k! \text{Gamma}[1] \text{Gamma}[m] \right.$$

$$\left. \text{Gamma}[1-1+m] \text{Hypergeometric2F1Regularized}\left[1, 1, 1+m, 1 - \frac{b^3}{a^3}\right]^2 + \right.$$

$$\left. (k-n)! (-1+n)! \left( a^3 + (-b^3 (-1+1) + a^3 (1-m)) \text{Gamma}[m] \right) \right.$$

$$\left. \left. \text{Hypergeometric2F1Regularized}\left[1, 1, 1+m, 1 - \frac{b^3}{a^3}\right] \right) \right)$$

---

## Export

In[19]:=

```
SetDirectory["D:\\Documents\\BA-Work\\scha21\\Sourcen"]  
Export["mathematicacode.pdf", EvaluationNotebook[]]
```

Out[19]= D:\Documents\BA-Work\scha21\Sourcen



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